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Flow Polynomials and Their Asymptotic Limits for Lattice Strip Graphs[1]

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We present exact calculations of flow polynomials $F(G, q)$ for lattice strips of various fixed widths $L_y \leqslant 4$ and arbitrarily great lengths L_x, with several different boundary conditions. Square, honeycomb, and triangular lattice strips are considered. We introduce the notion of flows per face fl in the infinite-length limit. We study the zeros of $F(G, q)$ in the complex q plane and determine exactly the asymptotic accumulation sets of these zeros $\mathscr{B}$ in the infinite-length limit for the various families of strips. The function fl is nonanalytic on this locus. The loci are found to be noncompact for many strip graphs with periodic (or twisted periodic) longitudinal boundary conditions, and compact for strips with free longitudinal boundary conditions. We also find the interesting feature that, aside from the trivial case $L_y = 1$, the maximal point, q_{cf}, where $\mathscr{B}$ crosses the real axis, is universal on cyclic and Möbius strips of the square lattice for all widths for which we have calculated it and is equal to the asymptotic value $q_{cf} = 3$ for the infinite square lattice.

KEY WORDS: Flow polynomial; Potts model; Tutte polynomial.

1. INTRODUCTION

An interesting problem in statistical mechanics is the enumeration of flows on bonds (edges) of a given graph satisfying certain conditions. In addition to its role in statistical physics interest, this problem is of interest in mathematics, engineering, business, and economics. Important questions in engineering and business concern the optimization of the flow capacity through parts of networks and the study of where bottlenecks in flows can occur because of limited connectivity. Here we shall focus on a particular

[1] Dedicated to F. Y. Wu on his 70th Birthday.

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set of discretized flows satisfying a certain conservation condition at each node of the network (i.e., vertex of the graph). The number of such flows is given by a certain flow polynomial. We shall present exact calculations of flow polynomials for a number of lattice strips with various widths L_y and arbitrarily great lengths L_x having several different types of boundary conditions. These exact calculations are valuable because, in general, the enumeration of flows on graphs takes an exponentially increasing time as a function of the number of vertices in the graph.

Consider a connected graph $G=(V, E)$ with vertex set V and edge (bond) set E and an abelian group H of order $o(H)$, represented as an additive group. For some general results, it will be necessary to allow the possibility of multiple edges joining a given pair of vertices, and loops (edges joining vertices to themselves), but these will usually not be present on the strip graphs of interest here. For definiteness, we let H be the additive group of integers mod q, $\mathbb{Z}_q$. Denote the number of vertices and edges as $n=|V|$ and $|E|$. Assign an orientation to each of the edges in G and consider a mapping that assigns to each of these oriented edges a nonzero element in $\mathbb{Z}_q$. A flow on G is then defined as an assignment of this type that satisfies the condition that the flow into each vertex is equal to the flow outward from this vertex. Here the addition of two flows is defined according to the abelian group H, so that, for $H=\mathbb{Z}_q$, the flow into each vertex is equal to the flow out of this vertex mod $\mathbb{Z}_q$. In a fluid or electric circuit analogy, this means that the (discretized) flow or electric current is conserved at each vertex mod q; there are no sinks or sources. Since the assignment of the zero element of $\mathbb{Z}_q$ to a given oriented edge is equivalent to the absence of the edge as far as the flow is concerned, a standard restriction is that an admissible flow must avoid zero flow numbers on any edge; this is termed a nowhere-zero q-flow. Henceforth, since all of our discussion will concern nowhere-zero q-flows, we shall take "q-flow" to mean "nowhere-zero q-flow." Some works discussing flow polynomials include refs. 1–24.

An important problem in mathematical graph theory concerns the enumeration of q-flows on a given (connected) graph G. Tutte showed that there exists a polynomial in q, which we denote $F(G, q)$, that equals this number of q-flows.[3, 5] The existence of this polynomial relies upon the fact that, for a given abelian group H, the number of flows on G depends only on the order of H, $o(H)=q$, not on any other structural properties of H. Clearly if G contains a bridge (isthmus), then there are no q-flows since the fluid or electric current flowing across the bridge has no way of returning. Thus, for any such edge that is a bridge, denoted e_b, one has $F(e_b, q)=0$. The minimum (integer) value of q such that a bridgeless graph G admits a q-flow is called the flow number of G, denoted $\phi(G)$.

We shall consider strips of several types of regular lattices, include the square (*sq*), honeycomb (*hc*), and triangular (*tri*) lattices. The longitudinal (transverse) direction is taken as the horizontal, x (vertical, y) direction. We envision the strip of the triangular lattice as being formed by starting with a strip of the square lattice and then adding edges joining, say, the lower left and upper right vertices of each square. The strips of the honeycomb lattice are envisioned as brick lattices. We use the symbols FBC_y and PBC_y for free and periodic transverse boundary conditions and FBC_x, PBC_x, and TPBC_x for free, periodic, and twisted periodic longitudinal boundary conditions. The term "twisted" means that the longitudinal ends of the strip are identified with reversed orientation. These strip graphs can be embedded on surfaces with the following topologies: (i) (FBC_y, FBC_x): free or open strip; (ii) (PBC_y, FBC_x): cylindrical; (iii) (FBC_y, PBC_x): cyclic; (iv) (FBC_y, TPBC_x): Möbius; (v) (PBC_y, PBC_x): torus; and (vi) (PBC_y, TPBC_x): Klein bottle.

We shall introduce the notion of flows per face, fl in the limit $|V| \to \infty$. From the basic definition one can generalize q from $\mathbb{Z}_+$ to $\mathbb{R}$ or, indeed, $\mathbb{C}$. This generalization is necessary when one calculates the zeros of $F(G, q)$ in the complex q plane. Using our exact calculations of $F(G, q)$ for a variety of families of lattice strip graphs, we shall determine exactly the continuous accumulation set of these zeros of $F(G, q)$ as $|V| \to \infty$ for each family of graphs G. This formal limit of the family of graphs G as $|V| \to \infty$ is denoted $\{G\}$. We shall call this accumulation set $\mathscr{B}$ or, when necessary to distinguish it from other accumulation sets, $\mathscr{B}_{fl}$. This locus is of interest because the function fl is nonanalytic across $\mathscr{B}$. We also study the approach of the fl functions for various infinite-length strips of the square lattice, as functions of q, to numerical values for the infinite square lattice.

In addition to its physical property as an enumeration of flows, the flow polynomial is also of physical interest because of a well-known property that for a planar graph G, it is equivalent (see Eq. (3.2) later) to the chromatic polynomial of the (planar) dual graph, $P(G^*, q)$, where $P(G, q)$ is defined as the number of ways of coloring the vertices of a graph G with q colors such that no two adjacent vertices have the same color. In turn, the chromatic polynomial is identical to the partition function of the zero-temperature q-state Potts antiferromagnet. We comment on the relation with previous calculations of chromatic polynomials (e.g., refs. 25–68). Given the connection (3.2), calculations of chromatic polynomials for families of planar lattice strip graphs immediately yield flow polynomials for the dual graphs. We shall not recapitulate these calculations here since they are available in the literature. Chromatic polynomials for nonplanar graphs cannot be immediately related to flow polynomials.

Besides the role of flows in statistical mechanics, there are several additional motivations for this work. The flow polynomial is of fundamental importance in graph theory. As we shall discuss later, it is a special case of the Tutte polynomial and encodes information on the connectivity of a graph. It is also of interest from the viewpoint of statistical mechanics because it is a special case of the q-state Potts model partition function. To our knowledge, aside from our recent brief discussion,[69] there has not been any study of the function fl that we introduce, and the associated regions of analyticity of fl separated by the locus $\mathscr{B}$ for strips of regular graphs. In particular, the points where $\mathscr{B}$ crosses or intersects the real q axis are somewhat analogous to phase transitions in statistical mechanics, in the sense that the fl function has different analytic forms on different sides of $\mathscr{B}$. Associated with this, one finds qualitatively different flow behavior, as described by fl, in the different regions separated by the locus $\mathscr{B}$. While the flow polynomial of a planar graph is equivalent to the chromatic polynomial of its dual graph, as we shall discuss further later, we find that the locus $\mathscr{B} \equiv \mathscr{B}_{fl}$ for the infinite-length limit of a strip graph of a regular lattice with periodic (or twisted periodic) longitudinal boundary conditions is rather different from the corresponding locus of zeros $\mathscr{B}_W$ of the chromatic polynomial for the infinite-length limit of this strip. These differences are especially intriguing in view of a theorem that we shall give later, showing that for the infinite two-dimensional lattice Λ, $\mathscr{B}_{fl}(\Lambda) = \mathscr{B}_W(\Lambda^*)$, where Λ^* is the planar lattice dual to Λ. In passing, we mention the "max-flow, min-cut" theorem of Ford and Fulkerson,[6] which is important for business and engineering applications. Here, we shall concentrate on intrinsic properties of flow polynomials and their connections with statistical mechanics.

2. CONNECTION WITH TUTTE POLYNOMIAL AND POTTS MODEL

In this section we review the connection between the flow polynomial and the Tutte polynomial or equivalently the Potts model partition function. Consider, as before, a connected graph $G = (V, E)$. If an edge of G is a loop, $e = e_\ell$, then the fluid (or electric current) conservation condition is automatically satisfied for any value of q, so that $F(e_\ell, q) = q - 1$, where for (nowhere-zero) flows the zero element of H is excluded and there are thus $q-1$ choices for the flow through the loop. We mention the standard notation that for an edge $e \in E$, G/e is the graph obtained from G by contraction on e, i.e., deleting the edge e and identifying the vertices that it joins, and $G - e$ is the graph obtained from G by deleting e. Then for an edge e that is not a bridge or a loop, the flow polynomial satisfies the deletion-contraction relation

$$F(G, q) = F(G/e, q) - F(G - e, q). \tag{2.1}$$

If G is a graph and K is a field, then a general function $g: G \to K$ is a Tutte–Gröthendieck invariant if (1) if $e \in E$ is a bridge, then $g(G) = g_b\, g(G/e)$, (2) if $e \in E$ is a loop, then $g(G) = g_\ell\, g(G/e)$, (3) if $e \in E$ is neither a bridge nor a loop, then

$$g(G) = ag(G-e) + bg(G/e), \qquad a, b \neq 0 \tag{2.2}$$

where g_b, g_ℓ, a, and b are independent of G (see Eq. (2.5) later). From the discussion above, it follows that $F(G, q)$ is a Tutte–Gröthendieck invariant. A useful property is that a Tutte–Gröthendieck invariant can be expressed in terms of the Tutte polynomial. A spanning subgraph $G' = (V, E')$ of a graph $G = (V, E)$ is a subgraph with the same vertex set and a subset $E' \subseteq E$ of the edge set of G. The Tutte polynomial of a graph G is defined as

$$T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G') - k(G)} (y-1)^{c(G')} \tag{2.3}$$

where $k(G')$ and $c(G')$ denote the number of components and linearly independent circuits of G', where, for an arbitrary graph $G = (V, E)$,

$$c(G) = |E| - |V| + k(G). \tag{2.4}$$

The quantity $c(G')$ is also called the co-rank or nullity of G', and the co-rank of the full graph G is called its cyclomatic number. The rank of a graph G is defined as $r(G) = |V| - k(G)$, so that the first exponent can be written equivalently as $k(G') - k(G) = r(G) - r(G')$. Since we deal only with connected graphs here, $k(G) = 1$. (We follow the standard notational usage of x and y for the arguments of the Tutte polynomial and caution the reader not to confuse these with the x and y directions along the strip graphs.)

A Tutte–Gröthendieck invariant g is then given as

$$g(G) = a^{|E| - |V| + 1} b^{|V| - 1} T\left(G, \frac{g_b}{b}, \frac{g_\ell}{a}\right). \tag{2.5}$$

In particular, for the flow polynomial, in terms of the notation above, we have $g_b = 0$, $g_\ell = q - 1$, $a = -1$, $b = 1$, whence

$$F(G, q) = (-1)^{|E| - |V| + 1}\, T(G, x = 0, y = 1 - q). \tag{2.6}$$

Thus, the flow polynomial is a special case of the Tutte polynomial. Hence, we can use our previous exact calculations of Tutte polynomials and Potts model partition functions for various families of graphs[69, 70] to derive flow polynomials for these graphs.

From (2.3) and (2.6), it follows that the degree of $F(G, q)$ as a polynomial in q is the cyclomatic number of G,

$$\deg F(G, q) = c(G). \tag{2.7}$$

As is clear from the fact that the cyclomatic number $c(G)$ gives the number of linearly independent circuits, it is closely related to the number of faces of G, $f(G)$. In particular, for a planar graph G, using the Euler relation $|V|-|E|+f(G)=2$, we have $f(G)=c(G)+1$.

Next, we recall the equivalence of the Tutte polynomial to the Potts model partition function. On a lattice, or more generally, a graph G, at temperature T, this model is defined by the partition function[10]

$$Z(G, q, v) = \sum_{\{\sigma_n\}} e^{-\beta \mathscr{H}} \tag{2.8}$$

with the Hamiltonian

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} \tag{2.9}$$

where $\sigma_i = 1,..., q$ are the spin variables on each vertex $i \in G$, and $\langle ij \rangle$ denotes pairs of adjacent vertices. We use the notation

$$K = \frac{J}{k_{\rm B} T}, \qquad v = e^K - 1 \tag{2.10}$$

(where there should not be any confusion between the variable v and the edge set V or between the variable K and the number of components of a graph $k(G)$) so that the physical ranges are (i) $v \geqslant 0$ corresponding to $\infty \geqslant T \geqslant 0$ for the Potts ferromagnet, and (ii) $-1 \leqslant v \leqslant 0$, corresponding to $0 \leqslant T \leqslant \infty$ for the Potts antiferromagnet.

As before, let $G' = (V, E')$ be a spanning subgraph of G. Then $Z(G, q, v)$ can be written as[77]

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')}. \tag{2.11}$$

This formula enables one to generalize q from $\mathbb{Z}_+$ to $\mathbb{R}$ or, indeed, $\mathbb{C}$. From it one also directly infers the equivalence

$$Z(G, q, v) = (x-1)^{k(G)} (y-1)^n T(G, x, y) \tag{2.12}$$

where

$$x = 1 + \frac{q}{v} \tag{2.13}$$

and

$$y = v + 1 \tag{2.14}$$

so that

$$q = (x-1)(y-1). \tag{2.15}$$

Combining (2.6) and (2.12), one has the relation

$$F(G, q) = (-1)^{|E|}\, q^{-n} Z(G, q, -q). \tag{2.16}$$

Thus, the flow polynomial is a special case of the q-state Potts model given, up to the above prefactor, by the evaluation

$$v = -q, \qquad \text{i.e.,} \quad K = \ln(1-q). \tag{2.17}$$

For the usual case of flows with $q \geqslant 2$, the condition $v = -q$ corresponds to a complex-temperature regime for the Potts model. In Eq. (2.17), one would thus have $K = \ln(q-1) + (2\ell+1)\, i\pi$, $\ell \in \mathbb{Z}$.

3. SOME GENERAL PROPERTIES OF FLOW POLYNOMIALS

3.1. Duality Relation with Chromatic Polynomial

Before presenting our new results, we recall some basic information about flow polynomials that will be relevant for our work. We first discuss a useful connection with chromatic polynomials. The chromatic polynomial $P(G, q)$ of a graph G counts the number of proper q-coloring of G, where a proper q-coloring is a coloring of the vertices of the graph using q colors, subject to the condition that the colors assigned to adjacent vertices are different. The minimum (positive integral) value of q that allows one to carry out a proper q-coloring of G is called the chromatic number of G, $\chi(G)$. The chromatic polynomial is given in terms of the Potts model partition function and Tutte polynomial as

$$P(G, q) = Z(G, q, v = -1) = (-1)^{n+1}\, q^{k(G)} T(G, x = 1-q, y = 0). \tag{3.1}$$

Physically, the chromatic polynomial is the special case of the partition function for the Potts antiferromagnet at zero temperature, $v=-1$. Now let G be a planar graph and G^* its planar dual graph. Then

$$F(G, q) = q^{-1}P(G^*, q). \tag{3.2}$$

This follows directly from the expressions for the flow and chromatic polynomials in terms of the Tutte polynomial, together with the symmetry relation of a Tutte polynomial

$$T(G, x, y) = T(G^*, y, x) \tag{3.3}$$

for a planar graph G. The equality (3.2) will be important for our later discussion. Several corollaries follow from this equality. First, let G be a bridgeless planar graph and G^* its planar dual, which thus has no loops. Then

$$\phi(G) = \chi(G^*). \tag{3.4}$$

(The restriction that G has no bridges is made so that flows exist on G, and this is dual to the restriction that G^* has no loops, so that proper q-colorings of G^* exist.)

3.2. Some Results on Existence of q-Flows

We mention here a few mathematical results on existence of q-flows that are relevant to our work. We recall the definition of the degree or coordination number of a vertex in a graph G as the number of edges connected to this vertex. An elementary theorem states that a bridgeless graph admits a 2-flow if and only if all of its vertex degrees are even. This can be understood as follows. After choosing an orientation for the edges, assign to each oriented edge the flow value 1. Since $1=-1 \bmod 2$, and since $2n=0 \bmod 2$, the fact that each vertex has even degree means that the flows into each vertex are $0 \bmod 2$. A related statement is that if a graph has flow number $\phi(G)=2$, then, since (for the nowhere-zero flows considered here) there is only one choice of flow number for each edge, namely 1, which is the same as $-1 \bmod 2$, it follows that $F(G, q=2)=1$.

One of the most important theorems pertaining to flow polynomials for planar graphs is the 4-flow theorem, which states that every bridgeless planar graph has a 4-flow. This is equivalent to the celebrated 4-color theorem,[78, 79] that every bridgeless planar graph has a coloring of faces with four colors such that any two faces that are adjacent across a given edge have different colors. An equivalent expression of the theorem for proper vertex colorings is that every loopless planar graph has a proper 4-coloring. One can inquire about the existence of flows for arbitrary, not necessarily planar, graphs. Jaeger proved that every bridgeless graph (not necessarily planar) has an 8-flow,[7] and subsequently Seymour proved the stronger result that every bridgeless graph has a 6-flow.[8] An outstanding conjecture, due to Tutte,[3] is that every bridgeless graph has a 5-flow. These general results provide a useful background for the specific flow numbers that we shall obtain for various planar and nonplanar families of graphs.

3.3. Structural Properties for General Strip Graphs

Since a flow polynomial is a special case of a Tutte polynomial or equivalently, a Potts model partition function, one can obtain several properties of flow polynomials from corresponding properties of the latter two polynomials. A general form for the Tutte polynomial for the strip graphs considered here, or more generally, for recursively defined families of graphs G_m comprised of m repeated subunits (e.g., the columns of squares of height L_y vertices that are repeated $L_x = m$ times to form strip of a regular lattice of width L_y and length L_x (denoted $L_y \times L_x$) with some specified boundary conditions), is[70]

$$T(G_m, x, y) = \frac{1}{x-1} \sum_{j=1}^{N_{T,G,\lambda}} c_{G,j} (\lambda_{T,G,j})^m \tag{3.5}$$

where the terms $\lambda_{T,G,j}$, the coefficients $c_{G,j}$, and the total number $N_{T,G,\lambda}$ depend on G through the type of lattice, its width, L_y, and the boundary conditions, but not on the length. Equivalently,

$$Z(G_m, q, v) = \sum_{j=1}^{N_{Z,G,\lambda}} c_{G,j} (\lambda_{Z,G,j})^m \tag{3.6}$$

where

$$N_{Z,G,\lambda} = N_{T,G,\lambda}. \tag{3.7}$$

It follows from (2.6) and (3.5) that the flow polynomial for recursive families of graphs, comprised of m repeated subgraph units, has the general form

$$F(G_m, q) = \sum_{j=1}^{N_{F,G,\lambda}} c_{G,j} (\lambda_{F,G,j})^m \tag{3.8}$$

where again the terms $\lambda_{F,G,j}$, the coefficients $c_{G,j}$, and the total number of terms $N_{F,G,\lambda}$ depend on G through the type of lattice, its width, L_y, and the boundary conditions, but not on the length. This is related to (3.5) as follows. The prefactor in (2.6), $(-1)^{|E|-|V|+1}$ is of the form $(-1)^{cm+1}$, where c is an even or odd integer. The single factor (-1) cancels the (-1) factor resulting from the $x=0$ evaluation of the overall prefactor $1/(x-1)$ in (3.5). For the cyclic lattice strips considered here, both $|V|$ and $|E|$ are integer multiples of $L_x=m$, so their difference is of the stated form, cm. Thus,

$$\lambda_{F,G,j}(q)=(-1)^c\,\lambda_{T,G,j}(x=0,\, y=1-q) \tag{3.9}$$

for the subset of the $\lambda_{T,G,j}$'s that are nonzero when evaluated at $x=0$.

For a given type of strip graph, the sum of the coefficients in (3.8) is denoted

$$C_{F,G}=\sum_{j=1}^{N_{F,G,\lambda}} c_{G,j}. \tag{3.10}$$

3.4. The Function *fl* and Associated Locus $\mathscr{B}$

Given the structural property (3.8), it is natural, in the limit $|V|\to\infty$, to define a function specifying the number of flows per face,

$$fl(\{G\},q)=\lim_{|V|\to\infty} F(G,q)^{1/f(G)}. \tag{3.11}$$

For almost all of the strip graphs considered here, $|V|\to\infty \Leftrightarrow f(G)\to\infty$. An exception is the circuit graph, for which $F(G)=2$, independent of $|V|$.

For reference, the reduced free energy is

$$f(\{G\},q,v)=\lim_{|V|\to\infty} \ln[Z(G,q,v)^{1/|V|}] \tag{3.12}$$

a limiting function obtained from the Tutte polynomial may be written as

$$\tau(\{G\},x,y)=\lim_{|V|\to\infty} T(G,x,y)^{1/|V|} \tag{3.13}$$

and the ground state degeneracy per vertex of the q-state Potts antiferromagnet is

$$W(\{G\},q)=\lim_{|V|\to\infty} P(G,q)^{1/|V|}. \tag{3.14}$$

Since $F(G, q)$ is a polynomial in q of degree $c(G)$, it follows that with this definition,

$$\lim_{q \to \infty} \frac{fl(\{G\}, q)}{q} = 1. \tag{3.15}$$

This is the analogue to the property that

$$\lim_{q \to \infty} \frac{W(\{G\}, q)}{q} = 1. \tag{3.16}$$

Following our earlier nomenclature,[37] we denote a term λ in (3.8) as leading (= dominant) for a given value of q if it has a magnitude greater than or equal to the magnitude of other λ's evaluated at this value of q. In the limit $|V| \to \infty$ the leading λ in $F(G_m, q)$ determines the function $fl(\{G\}, q)$. The continuous locus $\mathscr{B}$ where $fl(\{G\}, q)$ is nonanalytic thus occurs where there is a switching of dominant λ's in F and is the solution of the equation of degeneracy in magnitude of these dominant λ's. This is analogous to what happens for chromatic polynomials[28, 29] and, more generally for limits of recursively defined polynomials[30, 31] (see also ref. 32). Since a zero in $F(G, q)$ requires a cancellation between dominant λ's, it is clear that as $|V| \to \infty$, the continuous accumulation set $\mathscr{B}$ of the zeros of $F(G, q)$ forms the boundary curves across which this switching occurs and hence across which $fl(\{G\})$ is nonanalytic. Depending on the family of strip graphs, the locus $\mathscr{B}$ may or may not separate the complex q plane into different regions and may or may not cross the real q axis. If it does, we denote the maximal point where it crosses this axis as $q_{cf}(\{G\})$. We also denote the region including the positive real q axis extending down from $q = \infty$ (and terminating at $q_{cf}(\{G\})$ if the latter point exists) as region R_1. This region R_1 is understood to include the maximal area to which one can analytically continue $fl(\{G\})$ from the large-q positive real axis. For families of strip graphs where $\mathscr{B}$ separates the q plane into different regions, the function $fl(\{G\})$ has different analytic forms in the different regions. For a strip graph G, in region R_1,

$$fl(\{G\}, q) = \lambda_{R1,\, dom.} \tag{3.17}$$

where $\lambda_{R1,\, dom.}$ denotes the dominant λ in region R_1.

Just as was true for these functions f, τ, and W, there are two subtleties in the definition (3.11): (i) which of the $c(G)$ roots to take in the equation, and (ii) the fact that at certain values of q, one has the noncommutativity of limits

$$\lim_{|V| \to \infty} \lim_{q \to q_s} F(G, q)^{1/f(G)} \neq \lim_{q \to q_s} \lim_{|V| \to \infty} F(G, q)^{1/f(G)}. \tag{3.18}$$

We have discussed these before in the context of the definitions of W and f.[37, 52, 70] Concerning item (i), for sufficiently large real q, in the region R_1, $F(G, q)$ is real and positive, so one chooses the canonical root for $fl(\{G\})$, which is real and positive. However, if $\mathscr{B}$ separates the q plane into different regions, then in the latter regions, there is no canonical choice of phase that one can make for the root (3.11) and hence only the magnitude $|fl(\{G\}, q)|$ can be determined unambiguously. For item (ii) the noncommutativity typically occurs at special values $q_s = 1,..., \phi(G) - 1$. To avoid isolated discontinuities that would otherwise result where fl vanishes, we shall, following our earlier practice for W and f[37, 70] and adopt, for the cases where such noncommutativity occurs, the second order of limits in (3.18)

$$fl(\{G\}, q_s) = \lim_{q \to q_s} \lim_{|V| \to \infty} F(G, q)^{1/f(G)}. \tag{3.19}$$

From the duality relation (3.2) it follows that if G is a planar graph and G^* is its planar dual, then, taking into account that the number of faces of G is equal to the number of vertices of G^*, one has

$$fl(\{G\}, q) = W(\{G^*\}, q). \tag{3.20}$$

Note that if we had defined $fl(\{G\}, q)$ as $\lim_{|V| \to \infty} F(G, q)^{1/|V|}$, we would have obtained a different relation $fl(\{G\}, q) = W(\{G^*\}, q)^p$, where p is a geometric factor depending on the particular graphs G. Since the infinite square lattice (defined, as usual, as the limit of a finite $L_x \times L_y$ section of this lattice where $L_x \to \infty$, $L_y \to \infty$ with the ratio L_y/L_x equal to a finite nonzero constant) is self-dual, it follows that

$$fl(sq, q) = W(sq, q). \tag{3.21}$$

Since the infinite honeycomb and triangular lattices are the duals of each other, we also have

$$fl(tri, q) = W(hc, q), \qquad fl(hc, q) = W(tri, q). \tag{3.22}$$

The dual of the kagomé lattice is the diced lattice,[80, 81] and hence

$$fl(kag, q) = W(diced, q), \qquad fl(diced, q) = W(kag, q). \tag{3.23}$$

We next give some results for the locus $\mathcal{B}$. Because of the fact that these λ's are degenerate in magnitude on $\mathcal{B}$, it follows that $fl(\{G\}, q)$ is nonanalytic but continuous across $\mathcal{B}$. A basic property is that $\mathcal{B}(\{G\})$ is invariant under complex conjugation,

$$\mathcal{B}(\{G\}) = \mathcal{B}(\{G\})^*. \tag{3.24}$$

This is a consequence of the property that the coefficients of each term in the flow polynomial are real and hence the zeros of the flow polynomial are invariant under complex conjugation. Hence, the same property holds for their asymptotic accumulation set as $|V| \to \infty$.

We next point out another consequence of (3.2) and (3.20) for planar graphs. For this purpose, we must append subscripts to distinguish $\mathcal{B}_{fl}$ and $\mathcal{B}_W$, the respective continuous accumulation sets of zeros of the flow and chromatic polynomials in the limit $|V| \to \infty$. Then for the $|V| \to \infty$ limits of planar graphs $\{G\}$,

$$\mathcal{B}_{fl}(\{G\}) = \mathcal{B}_W(\{G^*\}). \tag{3.25}$$

Several corollaries follow. Since the infinite square lattice (defined, as usual, as the limit of a finite $L_x \times L_y$ section of this lattice where $L_x \to \infty$, $L_y \to \infty$ with the ratio L_y/L_x equal to a finite nonzero constant) is self-dual,

$$\mathcal{B}_{fl}(sq) = \mathcal{B}_W(sq). \tag{3.26}$$

Since the infinite triangular and honeycomb lattices are dual to each other, we have

$$\mathcal{B}_{fl}(tri) = \mathcal{B}_W(hc), \qquad \mathcal{B}_{fl}(hc) = \mathcal{B}_W(tri). \tag{3.27}$$

Since the kagomé and diced lattices[80, 81] are dual to each other,

$$\mathcal{B}_{fl}(kag) = \mathcal{B}_W(diced), \qquad \mathcal{B}_{fl}(diced) = \mathcal{B}_W(kag). \tag{3.28}$$

Now consider the lattice strip graphs $G[L_y \times L_x, cyc.]$ and $G[L_y \times L_x, Mb.]$. We have shown earlier[50, 52, 58] that the locus $\mathcal{B}_W$ is the same for the respective infinite-length limits of both of these strip graphs. We also find that the same property holds for the $\mathcal{B}_{fl}$ computed here, but note that this does not follow from our previous results for $\mathcal{B}_W$ since the Möbius strips are nonplanar, and hence one cannot apply the duality relation (3.2) and the consequent equalities (3.20) and (3.25). We also find that for the families of strip graphs that we have calculated, $\mathcal{B}_{fl}$ is the same for boundary conditions corresponding to embedding on surfaces with torus and Klein bottle topologies.

3.5. Noncompactness of $\mathscr{B}$ for Classes of Strip Graphs

An important feature of the loci $\mathscr{B} \equiv \mathscr{B}_{fl}$ for the infinite-length limits of the families of lattice strips with periodic (or twisted periodic) longitudinal boundary conditions is that many of these are noncompact in the q plane, containing curves that extend infinitely far away from the origin. *A fortiori*, this means that for this families of graphs, as $L_x \to \infty$, there is no upper bound on the magnitudes $|q|$ of the zeros of the flow polynomial. The noncompactness of $\mathscr{B}_{fl}$ for many strips of regular lattices is quite different from the loci $\mathscr{B}_W$ for the corresponding strips, which, as we showed in our previous works on chromatic polynomials, were compact in the q plane.[37–67] Sokal has proved that for an arbitrary graph G, any zero of the chromatic polynomial satisfies the following upper bound[53]

$$P(G, q) = 0 \;\Rightarrow |q| \leqslant c\Delta, \qquad c \simeq 7.964 \tag{3.29}$$

where here Δ denotes the maximal vertex degree in G. (Indeed, our explicit calculations for a variety of families of strip graphs yielded loci $\mathscr{B}_W$ on which the values of $\max(|q|)$ were considerably less than the above-mentioned upper bounds for the respective strips.) As was discussed in refs. 42–44, the locus $\mathscr{B}_W$ is noncompact if and only if the degeneracy condition in magnitude of two or more different dominant λ's can be satisfied for arbitrarily large $|q|$. Clearly, the same condition holds for $\mathscr{B}_{fl}$. One of us (RS), with Tsai, studied the conditions under which this could occur and constructed several general classes of families of graphs that were designed to satisfy this condition and thereby yield noncompact loci $\mathscr{B}_W$[37, 42–44] (see also refs. 35 and 53). As noted in refs. 42 and 43, the condition that $\lim_{|V| \to \infty} \Delta = \infty$ is a necessary but not sufficient condition for $\mathscr{B}_W$ to be noncompact; an example of a graph with a maximal vertex degree that goes to infinity as $|V| \to \infty$ but with a compact $\mathscr{B}_W$ is the wheel graph $K_1 + C_m$. Here, C_m denotes the circuit graph, K_p denotes the complete graph on p vertices, defined as the graph each of whose vertices is adjacent to every other vertex, and the "join" $G + H$ is defined as the graph formed by connecting each of the vertices of G to each of the vertices of H. (The $\mathscr{B}$ in this case is simply the unit circle $|q-2| = 1$.)

Using the duality relation (3.2) for planar graphs, one can understand why $\mathscr{B}_{fl}$ is noncompact for the simplest nontrivial cyclic strip of the square lattice, namely for width $L_y = 2$. To do this, we observe that the planar dual of this $2 \times L_x$ cyclic strip is the circuit graph with L_x vertices, augmented by having an extra vertex connected by edges to each of the vertices on the upper side of the strip and, separately, an extra vertex connected by edges to each of the vertices on the lower side of the strip. This is a special case of one of the families of graphs that were shown in refs. 42 and 43 to yield, in the $L_x \to \infty$ limit, noncompact loci $\mathscr{B}$, namely the family $(K_p)_b + G_r$, where, b means that b of the edges in the complete graph K_p are cut, and G_r denotes an arbitrary r-vertex graph. We refer the reader to Section 2 of ref. 42 and Section 2 of ref. 43 for further details. Specifically, we find that the planar dual graph to the $L_y = 2$ cyclic strip of the square lattice is the special case of $(K_p)_b + G_r$ with $p = 2$, $b = 1$, signifying the cutting of the single edge in the K_2, and $G_r = C_{L_x}$, the circuit graph. Using the duality relation (3.25), we therefore prove that $\mathscr{B}_{fl}$ for the $L_x \to \infty$ limit of the cyclic $L_y \times L_x$ strip is noncompact, extending infinitely far from the origin in the q plane. Similar arguments give insight into the noncompactness of $\mathscr{B}$ for the wider strips considered here. Because of the noncompactness of $\mathscr{B}_{fl}$ in the q plane and the property that we find that this locus does not pass through $q = 0$, it will often be convenient to display it in the plane of the inverse variable

$$u = \frac{1}{q} \tag{3.30}$$

where it is compact.

We note that not all strip graphs with periodic longitudinal boundary conditions lead to noncompact $\mathscr{B}_{fl}$. An example of a family with compact $\mathscr{B}_{fl}$ is provided by the cyclic self-dual strips of the square lattice $G_D(L_y \times L_x)$ that we studied in refs. 63 and 75. Since these are self-dual planar graphs, one can immediately use (3.25) to infer that the loci $\mathscr{B}_{fl}$ are identical to the loci $\mathscr{B}_W$ given in refs. 63 and 75, which are compact.

In contrast to the case with many families of strip graphs with periodic (or twisted periodic) longitudinal boundary conditions, we find that $\mathscr{B}$ is compact for the $L_x \to \infty$ limit of strips with free longitudinal boundary conditions, as will be illustrated with specific exact solutions later. This feature is similar to our previous findings for $\mathscr{B}_W$. Henceforth, where no confusion will result, we shall drop the subscript fl on $\mathscr{B}_{fl}$.

As in our previous work (e.g., ref. 70), we note that some aspects of the asymptotic accumulation sets of zeros of polynomials such as the Potts model partition function or one-variable special cases such as the chromatic polynomial or flow polynomial on the infinite-length limits of width-L_y lattice strips are not smooth functions of L_y; for example, the free energy of the full Potts model with partition function, $Z(G, q, v)$ has a ferromagnetic phase transition point only at zero temperature on an infinite-length, width L_y strip for any finite L_y, so that the limit of this phase transition temperature as $L_y \to \infty$ is also zero, whereas if one takes the thermodynamic limit $L_x \to \infty$, $L_y \to \infty$ with the ratio L_y/L_x equal to a finite constant, the model has a finite ferromagnetic phase transition temperature. This singularity at $T=0$, i.e., $K=\infty$, corresponds to the property that $\mathscr{B}$, plotted in the plane of the complex variable e^{-K}, for infinite-length cyclic strips[70–73] passes the origin $e^{-K}=0$. In contrast, since the Potts ferromagnet on (infinite) 2D lattices has a finite-temperature phase transition, the corresponding $\mathscr{B}$ does not pass through $e^{-K}=0$. We have remarked on similar noncommuting limits for $\mathscr{B}_W$ in our previous work, e.g., in our calculations of $\mathscr{B}_W$ for infinite-length strips of the triangular lattice with cyclic boundary conditions, we found that this locus always passes through $q=2$,[47, 48, 59] whereas, in contrast, the locus found in ref. 34 for the infinite-width limit of strips with cylindrical boundary conditions does not pass through $q=2$. Similarly, in our calculations of $\mathscr{B}_W$ for infinite-length strips of the square lattice with cyclic boundary conditions, we found that this locus always passes through $q=2$, whereas in our and other authors' calculations of $\mathscr{B}_W$ for infinite-length strips of the square lattice with cylindrical boundary conditions,[41, 55, 56, 60] it has been found that $\mathscr{B}$ does not pass through $q=2$, strongly suggesting that this difference will persist in the limit $L_y \to \infty$.

4. GENERAL STRUCTURAL RESULTS FOR CYCLIC STRIPS OF THE SQUARE AND HONEYCOMB LATTICES

In ref. 57 it was shown that for cyclic and Möbius strips of the square lattice of fixed width L_y and arbitrary length L_x (and also for cyclic strips of the triangular lattice) the coefficients c_j in the Tutte polynomial are polynomials in q with the property that for each degree d there is a unique polynomial, denoted $c^{(d)}$. Further, this was shown to be

$$c^{(d)} = U_{2d}(q^{1/2}/2) = \sum_{j=0}^{d} (-1)^j \binom{2d-j}{j} q^{d-j} \tag{4.1}$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. A number of properties of these coefficients were derived in ref. 57. We list below the specific $c^{(d)}$'s that will be needed here:

$$c^{(0)}=1, \qquad c^{(1)}=q-1, \qquad c^{(2)}=q^2-3q+1, \tag{4.2}$$

$$c^{(3)}=q^3-5q^2+6q-1. \tag{4.3}$$

Thus, the terms $\lambda_{T,L_y,j}$ that occur in (3.5) can be classified into sets, with the $\lambda_{T,L_y,j}(q,v)$ in the dth set being defined as those terms with coefficient $c^{(d)}$. In ref. 57 the numbers of such terms, denoted $n_T(L_y,d)$, were calculated. Labelling the eigenvalues with coefficient $c^{(d)}$ as $\lambda_{T,L_y,d,j}$ with $1 \leqslant j \leqslant n_T(L_y,d)$, the Tutte polynomial for a cyclic strip graph of length $L_x = m$ can be written in the form[57]

$$T(G_s[L_y \times m;\, cyc.], x, y) = \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_T(L_y,d)} (\lambda_{T,G_s,L_y,d,j})^m. \tag{4.4}$$

For the Möbius strip of the square lattice the coefficients may be either $+c^{(d)}$ or $-c^{(d)}$ and the terms are accordingly labelled as $\lambda_{T,L_y,d,\pm,j}$, where $1 \leqslant j \leqslant n_T(L_y,d,\pm)$. We have,[57] using the notation TPBC_y for twisted periodic b.c. in the longitudinal direction,

$$T(sq[L_y \times m;\, Mb.], x, y) = \frac{1}{x-1} \sum_{d=0}^{d_{\max}} c^{(d)} \left[\sum_{j=1}^{n_T(L_y,d,+)} (\lambda_{T,L_y,d,+,j})^m - \sum_{j=1}^{n_T(L_y,d,-)} (\lambda_{T,L_y,d,-,j})^m \right] \tag{4.5}$$

where

$$d_{\max} = \begin{cases} \frac{L_y}{2} & \text{for even } L_y, \\ \frac{L_y+1}{2} & \text{for odd } L_y. \end{cases} \tag{4.6}$$

The number $n_T(G_s,L_y,d)$ of λ's with a given coefficient $c^{(d)}$ is[57]

$$n_T(G_s,L_y,d) = \frac{(2d+1)}{(L_y+d+1)} \binom{2L_y}{L_y-d} \quad \text{for } G_s = sq, tri, hc; \quad 0 \leqslant d \leqslant L_y \tag{4.7}$$

and zero otherwise. The total number $N_{T,L_y,\lambda}$ of different terms $\lambda_{T,L_y,j}$ in Eq. (3.5) for cyclic (or Möbius) strips G_s of the square, triangular, and honeycomb lattices is[57]

$$N_{T,G_s,L_y,\lambda} = \sum_{d=0}^{L_y} n_T(G_s,L_y,d) \tag{4.8}$$

which was calculated to be[57, 73]

$$N_{T,G_s,L_y,\lambda} = \binom{2L_y}{L_y}. \tag{4.9}$$

For arbitrary L_y, Eq. (4.7) shows that there is a unique $\lambda_{T, L_y, d}$ corresponding to the coefficient $c^{(d)}$ of highest degree, $d = L_y$, and this term is

$$\lambda_{T, G_s, L_y, d = L_y} = 1. \tag{4.10}$$

Hence,

$$\lambda_{F, sq, L_y, d = L_y} = \lambda_{F, hc, L_y, d = L_y} = (-1)^{L_y + 1} \tag{4.11}$$

and

$$\lambda_{F, tri, L_y, d = L_y} = 1 \tag{4.12}$$

(independent of L_y). Since this eigenvalue is unique, it is not necessary to append another index, as with the other λ's, and we avoid this for simplicity.

From (4.4), it follows that the flow polynomial for a cyclic strip of the G_s-type lattice also has the same type of form, i.e.,

$$F(G_s[L_y \times m; cyc.], x, y) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_F(G_s, L_y, d)} (\lambda_{F, G_s, L_y, d, j})^m. \tag{4.13}$$

From our earlier work on Tutte polynomials, we found that for a strip of a given lattice type and width, the same set of λ's occurs for the case of cyclic and Möbius boundary conditions, and this implies that the same is true of the flow polynomial. It follows that for a strip of a given lattice type and width, in the infinite-length limit, the accumulation sets are the same for cyclic and Möbius boundary conditions. Let us denote the number of $\lambda_{F, G_s, L_y, d, j}$ with coefficient $c^{(d)}$ as $n_F(G_s, L_y, d)$. Then for this strip

$$C_{F, G_s, L_y} = \sum_{d=0}^{L_y} n_F(G_s, L_y, d)\, c^{(d)}. \tag{4.14}$$

We now concentrate on cyclic strips of the square and honeycomb lattice. Aside from degenerate cases, the dual of the cyclic strip of the square lattice of width $L_y \geqslant 2$ and length L_x is a strip of the same lattice with width $L_y - 1$ and length L_x with all of the vertices along the upper and lower sides connected to two respective external vertices. Now consider proper colorings of this dual graph. The total number of proper q-colorings of a transverse slice is $q(q-1)^{L_y}$. It is elementary to show that this also holds for $L_y = 1$. From (3.2), it follows that

$$C_{F, sq, L_y} = (q-1)^{L_y}. \tag{4.15}$$

A similar argument shows that (4.15) also holds for cyclic strips of the honeycomb lattice.

We now derive the following structural theorem.

Theorem 4.1. Consider the flow polynomial for a cyclic strip graph G_s of the square or honeycomb lattice of fixed width L_y and arbitrarily great length L_x. For brevity, set $n_F(G_s, L_y, d) \equiv n_F(L_y, d)$. The $n_F(L_y, d)$, $d = 0, 1,..., L_y$ are determined as follows. One has

$$n_F(L_y, d) = 0 \qquad \text{for} \quad d > L_y, \tag{4.16}$$

$$n_F(L_y, L_y) = 1 \tag{4.17}$$

$$n_F(1, 0) = 0 \tag{4.18}$$

with all other numbers $n_F(L_y, d)$ being determined by the two recursion relations

$$n_F(L_y+1, 0) = n_F(L_y, 1) \tag{4.19}$$

and

$$n_F(L_y+1, d) = n_F(L_y, d-1) + n_F(L_y, d) + n_F(L_y, d+1)$$
$$\text{for} \quad L_y \geqslant 1 \quad \text{and} \quad 1 \leqslant d \leqslant L_y+1. \tag{4.20}$$

Proof. We substitute for $c^{(d)}$ from Eq. (4.1) in Eq. (4.15). We obtain another equation by differentiating this with respect to q once; another by differentiating twice, and so forth up to L_y-fold differentiations. This yields L_y+1 linear equations in the L_y+1 unknowns, $n_F(L_y, d)$, $d = 0, 1,..., L_y$. We solve this set of equations to get the $n_F(L_y, d)$. ∎

Corollary 4.1.

$$n_F(L_y, L_y-1) = L_y - 1. \tag{4.21}$$

We note that the recursion relations (4.19), (4.20) for the numbers $n_F(L_y, d)$ for cyclic strips of the square and honeycomb lattices are the same as the recursion relations that we derived earlier in Eqs. (3.14) and (3.15) of ref. 57 for the corresponding numbers $n_P(L_y, d)$ for cyclic strips of the square and triangular lattice. Thus, the differences in the values of $n_F(L_y, d)$ for cyclic strips of the square and honeycomb lattice and the $n_P(L_y, d)$ for cyclic strips of the square and triangular lattice are due to the different initial values of these quantities, i.e., (4.18) in the former case and $n_P(1, 0) = n_P(1, 1) = 1$ in the latter case.

Table I. Table of Numbers $n_F(L_y, d)$ and Their Sums, $N_{F,L_y,\lambda}$ for Cyclic Strips of the Square and Honeycomb Lattices. Blank Entries Are Zero

	d											
L_y	0	1	2	3	4	5	6	7	8	9	10	$N_{F,L_y,\lambda}$
1	0	1										1
2	1	1	1									3
3	1	3	2	1								7
4	3	6	6	3	1							19
5	6	15	15	10	4	1						51
6	15	36	40	29	15	5	1					141
7	36	91	105	84	49	21	6	1				393
8	91	232	280	238	154	76	28	7	1			1107
9	232	603	750	672	468	258	111	36	8	1		3139
10	603	1585	2025	1890	1398	837	405	155	45	9	1	8953

Corollary 4.2.

$$n_F(L_y,0)=\frac{1}{L_y+1}\sum_{j=1}^{[(L_y+1)/2]}\binom{L_y+1}{j}\binom{L_y-j-1}{j-1}\qquad\text{for}\quad L_y\geqslant 2. \tag{4.22}$$

Proof. This follows immediately from the solution to our general recursion relations (4.19), (4.20). ∎

Summing the $n_F(L_y, d)$ over d for a given strip with L_y, we obtain

$$N_{F,L_y,\lambda}=\sum_{j=0}^{[L_y/2]}\binom{L_y}{j}\binom{L_y-j}{j} \tag{4.23}$$

where $[x]$ is integer part of x. This total number also applies to Möbius strips of the square and honeycomb lattice.

A generating function for the $N_{F,L_y,\lambda}$ is[82, 83]

$$\frac{1}{\sqrt{1-2x-3x^2}}-1=\sum_{L_y=1}^{\infty} N_{F,L_y,\lambda}x^{L_y}. \tag{4.24}$$

From this, it follows that the number $N_{F,L_y,\lambda}$ grows exponentially fast with the width L_y of the cyclic strip of the square or honeycomb lattice, with the leading asymptotic behavior

$$N_{F,L_y,\lambda}\sim L_y^{-1/2}3^{L_y}\qquad\text{as}\quad L_y\to\infty. \tag{4.25}$$

Analogous structural results can also be given for cyclic strips of the triangular lattice. We proceed to present exact calculations of flow polynomials for a number of different lattice strips.

5. CYCLIC AND MÖBIUS STRIPS OF THE SQUARE LATTICE

5.1. General

We denote the cyclic family as $sq(L_y, L_x, \mathrm{FBC}_x, \mathrm{PBC}_y) \equiv sq(L_y, L_x, cyc.)$ and the Möbius family as $sq(L_y, L_x, \mathrm{FBC}_x, \mathrm{TPBC}_y) \equiv sq(L_y, L_x, Mb.)$. The cyclic and Möbius strips of the square lattice of width L_y and length L_x have $|V| = L_x L_y$ vertices and $|E| = L_x(2L_y - 1)$, so that the prefactor in (2.6) is $(-1)^{L_x(L_y-1)+1}$ and hence the $\lambda_{F,sq,L_y,d,j}$ are given by the subset of the $\lambda_{T,sq,L_y,d,j}$'s that are nonzero when evaluated for $x = 0$ and $y = 1 - q$, multiplied by the prefactor $(-1)^{L_y-1}$. Among these families of graphs, the lowest case, $L_y = 1$, is just the circuit graph C_m, and the result is elementary; the only $\lambda_{F,sq,j}$ is unity, and $F(C_m, q) = q - 1$ (independent of m). This polynomial has only a single zero, at $q = 1$. With the definition (3.11), we obtain $fl(q) = \sqrt{q-1}$. Henceforth, for brevity of notation, where no confusion will result, we shall omit the F in $\lambda_{F,sq,L_y,d,j}$ and write this simply as $\lambda_{sq,L_y,d,j}$, and similarly for other lattice types.

5.2. $L_y = 2$ Cyclic and Möbius Strips of the Square Lattice

Using the duality relation (3.2), one has

$$F(sq[2 \times L_x = m, cyc.], q) = q^{-1} P((K_2)_1 + C_m, q) \tag{5.1}$$

where K_p is the complete graph, $G + H$ is the join of the graphs G and H, $(K_p)_b$ denotes the graph obtained by removing b edges from K_p, and C_m is the circuit graph with m vertices. In refs. 37, 42, 43, one of us (R.S.), with Tsai, calculated the chromatic polynomial for the family $(K_2)_1 + C_m$. This immediately gives the flow polynomial, via Eq. (5.1), with

$$\lambda_{sq,2,0,1} = q - 2 \tag{5.2}$$

$$\lambda_{sq,2,1,1} = q - 3 \tag{5.3}$$

$$\lambda_{sq,2,2} = -1. \tag{5.4}$$

Hence, we find

$$F(sq[2 \times m, cyc.], q) = (q-2)^m + c^{(1)}(q-3)^m + c^{(2)}(-1)^m. \tag{5.5}$$

Note that $n_F(2,0)=n_F(2,1)=n_F(2,2)=1$ and $N_{F,2,\lambda}=3$, in accord with our general structural formulas given above.

By using results that were obtained in ref. 70 for the Tutte polynomial, we obtain

$$F(sq[2\times m,\, Mb.],q)=(q-2)^m+c^{(1)}(q-3)^m-(-1)^m. \tag{5.6}$$

Thus,

$$\begin{aligned} &n_F(sq,2,0,+)=1, \qquad n_F(sq,2,0,-)=1, \\ &n_F(sq,2,1,+)=1, \qquad n_F(sq,2,1,-)=0 \end{aligned} \tag{5.7}$$

with $n_F(sq,2,d,\pm)=0$ for $d\geqslant 2$.

We observe that $F(sq[L_y=2,L_x=m,cyc.],q)$ and $F(sq[L_y=2, L_x=m,\, Mb.],q)$ always have the factors $(q-1)(q-2)$. For $m\geqslant 3$ odd, $F(sq[L_y=2,L_x=m,cyc.],q)$ also has the factor $(q-3)$, while for m even, $F(sq[L_y=2,L_x=m,\, Mb.],q)$ also has the factor $(q-3)$. These results show that

$$\phi(sq[2\times m,\, cyc.])=\begin{cases} 3 & \text{if } m \text{ is even} \\ 4 & \text{if } m\geqslant 3 \text{ is odd} \end{cases} \tag{5.8}$$

(for $m=1$, this flow polynomial vanishes since the graph contains a bridge). Further,

$$\phi(sq[2\times m,\, Mb.])=\begin{cases} 4 & \text{if } m \text{ is even} \\ 3 & \text{if } m \text{ is odd.} \end{cases} \tag{5.9}$$

We recall that a graph G is k-critical if $P(G,q=\chi(G))=k!$, where the chromatic number of G, $\chi(G)$, was defined above. Somewhat analogously to the concept of k-critical graphs for proper vertex coloring and chromatic polynomials, one may ask whether for a graph G with $\phi(G)=k$ it is true that $F(G,q=\phi(G))$ has a fixed value depending on q, i.e., for $q=\phi(G)$, the number of flows on G is a fixed number rather than growing (exponentially) with $|V|$. Using our results we can answer the question for these cyclic and Möbius strip graphs. First, we recall the elementary observation that if $\phi(G)=2$, then $F(G,2)=1$. From Eqs. (5.8) and (5.9) we know that the flow number is 3 for the $L_y=2$ cyclic and Möbius strips of the square lattice with even and odd $L_x=m$, respectively, and we find

$$F(sq[2\times m,\, cyc.],q=3)=2 \qquad \text{for even} \quad m\geqslant 2 \tag{5.10}$$

and

$$F(sq[2 \times m, Mb.], q=3) = 2 \qquad \text{for odd} \quad m \geqslant 1. \tag{5.11}$$

In contrast, it is readily verified from our exact results that for the $L_y = 2$ cyclic and Möbius strips of the square lattice with odd and even $L_x = m$, respectively, for which $\phi = 4$, the evaluation of the respective flow polynomials at $q = 4$ yields values that grow (exponentially) with the length of the strip.

One way to prove Eqs. (5.10) and (5.11) is to use our explicit calculations of the flow polynomials for these families of graphs. Another way is to write down the actual flows. For this purpose, consider a given square on the strip. For $q = 3$ there are two circular flows on this square, namely those with a flow number 1 assigned to each edge, going in (i) a clockwise or (ii) a counterclockwise manner. Clearly, a flow with flow number 1 assigned to each edge, going in a clockwise manner is equivalent to a flow with flow number $2 = -1 \bmod 3$ assigned to each edge going in a counterclockwise manner. Consider the flow of type (i) on this square. In order to satisfy the flow conservation condition at the right-hand upper and lower vertices, it is necessary that the flow on the neighboring square to the right be of type (ii). Similarly, if the flow on the given square is of type (ii), then it is necessary that the flow on the neighboring square to the right be of type (i). Continuing in this manner along the cyclic strip, one finds a consistent set of choices if m is even, but not if m is odd. This proves the first part of the corollary. A similar proof with obvious changes works for the second part.

We remark that the constructive proofs given above establish that for $q = 3$ the total flows can be decomposed into superpositions of circulations around each square. Since there is a 1–1 correspondence with the flows around a face of a graph and the face-colorings of the graph,(3) one sees that these corollaries are equivalent to an analogous condition for the face-coloring of the strip graph, or equivalently, the vertex coloring of the dual graph.

In accordance with our general discussion given above, the locus $\mathscr{B}$ for the $L_x \to \infty$ limit of the cyclic and Möbius $L_y = 2$ strips of the square lattice is noncompact in the q plane. From Eq. (5.1), it follows that this locus is identical to the one that was determined earlier for the $m \to \infty$ limit of the family $P((K_2)_1 + C_m, q)$ in refs. 37, 42, 43 and shown in the q plane shown in Fig. 2 of ref. 37, or equivalently, to the locus $\mathscr{B}$ in the $u = 1/q$ plane shown in Fig. 1 of ref. 43. It divides the q plane into three regions, R_i, $i = 1, 2, 3$. Regions R_1, R_2, and R_3 contain the respective real intervals

$3 \leqslant q \leqslant \infty$; $2 \leqslant q \leqslant 3$, and $q \leqslant 2$. Region R_2 is bounded on the left by the arc of the circle

$$q = 3 + e^{i\theta}, \qquad \text{for} \quad \frac{2\pi i}{3} \leqslant \theta \leqslant \frac{4\pi i}{3} \tag{5.12}$$

and on the right by the arc of the circle

$$q = 2 + e^{i\theta}, \qquad \text{for} \quad -\frac{\pi i}{3} \leqslant \theta \leqslant \frac{\pi i}{3}. \tag{5.13}$$

Thus,

$$q_{cf} = 3 \qquad \text{for} \quad \{G\} = sq,\, 2 \times \infty,\, cyc./Mb. \tag{5.14}$$

The two arcs (5.12) and (5.13) meet at the complex-conjugate points

$$q_t, q_t^* = \frac{5}{2} \pm \frac{\sqrt{3}\, i}{2} = 2 + e^{\pm i\pi/3} \tag{5.15}$$

which are triple points on the locus $\mathscr{B}$. Curves on $\mathscr{B}$ extend upwards from q_t and downwards from q_t^* separating region R_1 from R_3. The curves on $\mathscr{B}$ pass through the origin $u = 0$ vertically. This follows because at $u = 0$ there are two λ's which are leading and are degenerate in magnitude.

In region R_1, $\lambda_{sq,2,0,1}$ is dominant, so that

$$fl(sq[2 \times \infty, cyc.], q) = fl(sq[2 \times \infty, Mb.], q) = q - 2 \qquad \text{for} \quad q \in R_1. \tag{5.16}$$

In region R_2, $\lambda_{sq,2,2}$ is dominant, so that

$$|fl(sq[2 \times \infty, cyc./Mb.], q)| = 1 \qquad \text{for} \quad q \in R_2. \tag{5.17}$$

In region R_3, $\lambda_{sq,2,1,1}$ is dominant, so that

$$|fl(sq[2 \times \infty, cyc./Mb.], q)| = |q - 3| \qquad \text{for} \quad q \in R_3. \tag{5.18}$$

5.3. $L_y = 3$ Cyclic and Möbius Strips of the Square Lattice

Our general structural results yield $n_F(3, 0) = 1$, $n_F(3, 1) = 3$, $n_F(3, 2) = 2$, $n_F(3, 3) = 1$ with the total number $N_{F,3,\lambda} = 7$. For this $L_y = 3$ case we find that a single term can occur with more than one degree for its coefficient; in particular, the term $(2 - q)$ occurs with both coefficient $c^{(1)}$ and coefficient $c^{(2)}$. Given that this phenomenon occurs, the sum (4.23) is not, in general, identical to the sum of distinct λ's but instead is an upper bound on this sum. This is a type of behavior that does not occur for the Tutte or chromatic polynomials of strips of regular planar lattices that we have considered in previous work. For these strips, the sums of the corresponding numbers $n_T(L_y, d)$ and $n_P(L_y, d)$ over d for a given value of L_y are equal to the respective numbers of distinct λ's in the Tutte and chromatic polynomials. From our general calculation of the Tutte polynomial in ref. 71, we find that the nonzero terms are

$$\lambda_{sq,3,0,1} = q^2 - 5q + 7 \tag{5.19}$$

$$\lambda_{sq,3,1,1} = 2 - q \tag{5.20}$$

$$\lambda_{sq,3,1,j} = \tfrac{1}{2}[10 - 6q + q^2 \pm (52 - 56q + 28q^2 - 8q^3 + q^4)^{1/2}] \quad \text{for} \quad j = 2, 3 \tag{5.21}$$

$$\lambda_{sq,3,2,1} = 2 - q \tag{5.22}$$

$$\lambda_{sq,3,2,2} = 4 - q \tag{5.23}$$

$$\lambda_{sq,3,3} = 1. \tag{5.24}$$

We label the terms in the flow polynomial for the Möbius strip as $\lambda_{sq,L_y,d,\pm,j}$ where d, $\pm$ means that the term has the coefficient $\pm c^{(d)}$. For the terms $\lambda_{sq,L_y,d,\pm,j}$ occurring in the flow polynomial for the Möbius strip with width $L_y = 3$ we find

$$\lambda_{sq,3,0,+,1} = \lambda_{sq,3,0,1} = q^2 - 5q + 7 \tag{5.25}$$

$$\lambda_{sq,3,0,+,2} = \lambda_{sq,3,2,1} = 2 - q \tag{5.26}$$

$$\lambda_{sq,3,0,-,1} = \lambda_{sq,3,2,2} = 4 - q \tag{5.27}$$

$$\lambda_{sq,3,1,+,j} = \lambda_{sq,3,1,j+1} = \tfrac{1}{2}[10 - 6q + q^2 \pm (52 - 56q + 28q^2 - 8q^3 + q^4)^{1/2}] \quad \text{for} \quad j = 1, 2 \tag{5.28}$$

$$\lambda_{sq,3,1,-,1} = \lambda_{sq,3,1,1} = 2 - q \tag{5.29}$$

$$\lambda_{sq,3,2,+,1} = \lambda_{sq,3,3} = 1. \tag{5.30}$$

Then

$$F(sq[3 \times m, cyc.], q) = (\lambda_{sq,3,0,1})^m + c^{(1)} \sum_{j=1}^{3} (\lambda_{sq,3,1,j})^m + c^{(2)} \sum_{j=1}^{2} (\lambda_{sq,3,2,j})^m + c^{(3)} \tag{5.31}$$

and

$$F(sq[3\times m, Mb.], q)$$
$$= (\lambda_{sq,3,0,+,1})^m + (\lambda_{sq,3,0,+,2})^m - (\lambda_{sq,3,0,-,1})^m$$
$$+ c^{(1)}((\lambda_{sq,3,1,+,1})^m + (\lambda_{sq,3,1,+,2})^m - (\lambda_{sq,3,1,-,1})^m) + c^{(2)}. \tag{5.32}$$

We observe that $F(sq[3\times m, cyc.], q)$ and $F(sq[3\times m, Mb.], q)$ always have the factors $(q-1)(q-2)$. For $m \geqslant 3$ odd, $F(sq[3\times m, cyc.], q)$ also has the factor $(q-3)$. The flow numbers for the cyclic $3\times m$ strip are therefore the same as for the cyclic $2\times m$ strip of the square lattice, given above in Eq. (5.8). Further, we have

$$\phi(sq[3\times m, Mb.]) = 3. \tag{5.33}$$

We also find

$$F(sq[3\times m, cyc.], q=3) = \begin{cases} 6 & \text{if } m \geqslant 2 \text{ is even} \\ 0 & \text{if } m \geqslant 1 \text{ is odd} \end{cases} \tag{5.34}$$

$$F(sq[3\times m, Mb.], q=3) = \begin{cases} 2 & \text{if } m \geqslant 2 \text{ is even} \\ 4 & \text{if } m \geqslant 1 \text{ is odd.} \end{cases} \tag{5.35}$$

The locus $\mathscr{B}$ for $L_x \to \infty$ is shown in Fig. 1. Again, this locus is noncompact in the q plane. The locus is shown in the u plane, where it is compact, in Fig. 2. The locus separates the q plane into several regions. Three of these regions, R_j, $j = 1, 2, 3$, contain intervals of the real axis, which are $q \geqslant 3$ for R_1, $2 \leqslant q \leqslant 3$ for R_2, and $q < 2$ for R_3. Hence,

$$q_{cf} = 3 \qquad \text{for} \quad \{G\} = sq, 3\times\infty, cyc./Mb. \tag{5.36}$$

A general feature that we find is that, aside from the trivial case $L_y = 1$, for which $\mathscr{B} = \varnothing$, $\mathscr{B}$ crosses the real axis at $q = 2$ and $q = 3$ for all of the widths $L_y = 2, 3, 4$ for which we have obtained exact general formulas for the flow polynomials (see later for the $L_y = 4$ case). For $L_y = 3$, the dominant terms in the above-mentioned three regions are (with an appropriate choice of branch cut for the square root) $\lambda_{sq,3,1,2}$ in R_1, $\lambda_{sq,3,2,2}$ in R_2, and $\lambda_{sq,3,1,2}$ in R_3, so that

$$fl(sq[3\times\infty, cyc./Mb.], q) = (\lambda_{sq,3,1,2})^{1/2} \qquad \text{for} \quad q \in R_1. \tag{5.37}$$

In region R_2, $\lambda_{sq,3,2,2}$ is dominant, so that

$$|fl(sq[3\times\infty, cyc./Mb.], q)| = |q-4|^{1/2} \qquad \text{for} \quad q \in R_2. \tag{5.38}$$

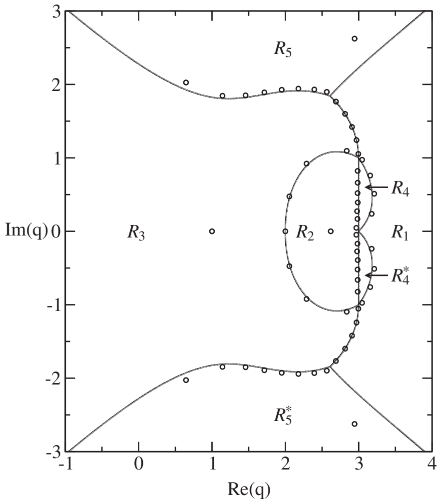

Fig. 1. Singular locus $\mathscr{B}$ in the q plane for $fl(sq, 3\times\infty, cyc./Mb., q)$ for the $3\times\infty$ strip of the square lattice with cyclic or Möbius ($Mb.$) boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 3\times L_x, cyc., q)$ for $L_x = 30$ (so that this polynomial has degree equal to $c(G) = 61$) are also shown.

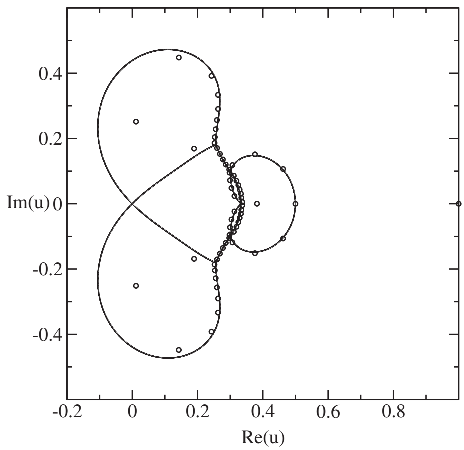

Fig. 2. Singular locus $\mathscr{B}$ in the u plane for $fl(sq, 3\times\infty, cyc./Mb., q)$ for the $3\times\infty$ strip of the square lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 3\times L_x, cyc., q)$ for $L_x = 30$ are also shown.

In region R_3, $\lambda_{sq,3,1,2}$ is dominant, so that

$$|f(sq[3\times\infty, cyc./Mb.], q)| = |\lambda_{sq,3,1,2}|^{1/2} \qquad \text{for} \quad q \in R_3. \tag{5.39}$$

There is also a pair of small complex-conjugate enclosed regions located to the upper and lower right of the point $q=3$; in these regions, $\lambda_{sq,3,2,1}$ is dominant. The part of $\mathscr{B}$ forming the boundary between region R_2 and regions R_4, R_4^* is the vertical line segment given by $Re(q)=3$, $-1 \leqslant Im(q) \leqslant 1$. Four curves on $\mathscr{B}$ meet at the intersection points $q=\pm i$, and as one moves further upward above $q=i$, $\mathscr{B}$ consists of a boundary separating regions R_1 and R_3. This boundary bifurcates at $q \simeq 2.6+1.8i$, and as one moves to larger values of $Im(q)$, one enters the region R_5. Correspondingly, there is the complex-conjugate region R_5^*. Expanding the equation for the degeneracy of magnitudes of leading terms λ around $u=0$, we have

$$|1-5u+7u^2| = |1-5u+8u^2+O(u^3)|. \tag{5.40}$$

Expressing this in terms of polar coordinates, with $u=re^{i\theta}$, and expanding around $r=0$, we get the equation, to leading order in r,

$$r^2 \cos 2\theta = 0 \qquad \text{as} \quad r \to 0. \tag{5.41}$$

The solution to this equation is

$$\theta = \frac{(2j+1)\,\pi}{4}, \qquad j=0, 1, 2, 3 \tag{5.42}$$

i.e., $\theta = \pm\pi/4, \pm 3\pi/4$, so that the curves approach $u=0$ with these angles.

5.4. $L_y = 4$ Cyclic and Möbius Strips of the Square Lattice

For the flow polynomial of the cyclic $4\times m$ strip of the square lattice our general results yield $n_F(4,0)=3$, $n_F(4,1)=6$, $n_F(4,2)=6$, $n_F(4,3)=3$, $n_F(4,4)=1$ with the total number $N_{F,4,\lambda}=19$. We have calculated the flow polynomial for this case. We find that the three $\lambda_{F,4,0,j}$, $j=1,2,3$, with coefficients $c^{(0)}$ are the roots of the equation

$$\xi^3-(q^3-8q^2+24q-26)\,\xi^2 \\ -(q^5-12q^4+59q^3-149q^2+193q-101)\,\xi+(q-3)(q-2)^5=0. \tag{5.43}$$

(Here $\lambda_{F,4,0,j} \equiv \lambda_{sq,4,0,j}$, etc.)

For the λ's with coefficient $c^{(1)}$, we have

$$\lambda_{F,4,1,1} = -(q-2)^2 \tag{5.44}$$

$$\lambda_{F,4,1,2} = -(q-3)^2. \tag{5.45}$$

The $\lambda_{F,4,1,j}$ for $j=3, 4, 5, 6$ are the roots of the equation

$$\begin{aligned}\xi^4 &- (q-3)(q^2-6q+12)\,\xi^3 - (q-2)(2q^4-22q^3+96q^2-196q+159)\,\xi^2 \\ &- (q-3)(q^6-16q^5+105q^4-366q^3+717q^2-750q+327)\,\xi \\ &+ (q^6-15q^5+93q^4-306q^3+565q^2-555q+227)(q-2)^2 = 0.\end{aligned} \tag{5.46}$$

For the λ's with coefficient $c^{(2)}$,

$$\lambda_{F,4,2,j} = \tfrac{1}{2}\left(-(q-3)^2 \pm \sqrt{(q-3)(q^3-5q^2+11q-11)}\right) \qquad \text{for} \quad j=1, 2. \tag{5.47}$$

The $\lambda_{F,4,2,j}$ for $j=3, 4, 5, 6$ are roots of the equation

$$\begin{aligned}\xi^4 &+ (2q^2-13q+22)\,\xi^3 + (q-2)(q^3-13q^2+51q-66)\,\xi^2 \\ &- (2q^5-28q^4+152q^3-402q^2+521q-265)\,\xi \\ &+ (q^4-11q^3+43q^2-70q+41)(q-2)^2 = 0.\end{aligned} \tag{5.48}$$

For the λ's with coefficient $c^{(3)}$ we find

$$\lambda_{F,4,3,1} = q-3 \tag{5.49}$$

$$\lambda_{F,4,3,2} = q-(3-\sqrt{2}) \tag{5.50}$$

$$\lambda_{F,4,3,3} = q-(3+\sqrt{2}) \tag{5.51}$$

and, in accordance with Eq. (4.11),

$$\lambda_{F,4,4} = -1. \tag{5.52}$$

Hence

$$\begin{aligned}F(sq[4\times m, cyc.], q) &= \sum_{j=1}^{3} (\lambda_{sq,4,0,j})^m + c^{(1)} \sum_{j=1}^{6} (\lambda_{sq,4,1,j})^m \\ &\quad + c^{(2)} \sum_{j=1}^{6} (\lambda_{sq,4,2,j})^m + c^{(3)} \sum_{j=1}^{3} (\lambda_{sq,4,3,j})^m + c^{(4)}(-1)^m.\end{aligned} \tag{5.53}$$

For the Möbius strip of the square lattice with width $L_y = 4$ we find $n_F(4, 0, +) = 5$, $n_F(4, 0, -) = 4$, $n_F(4, 1, +) = 4$, $n_F(4, 1, -) = 3$, $n_F(4, 2, +) = 2$, $n_F(4, 2, -) = 1$, with

$$\lambda_{sq,4,0,+,j} = \lambda_{sq,4,0,j}, \qquad j = 1, 2, 3 \tag{5.54}$$

$$\lambda_{sq,4,0,+,4} = \lambda_{sq,4,2,1}, \qquad \lambda_{sq,4,0,+,5} = \lambda_{sq,4,2,2} \tag{5.55}$$

$$\lambda_{sq,4,0,-,j} = \lambda_{sq,4,2,j+2}, \qquad j = 1, 2, 3, 4 \tag{5.56}$$

$$\lambda_{sq,4,1,+,j} = \lambda_{sq,4,1,j+2}, \qquad j = 1, 2, 3, 4 \tag{5.57}$$

$$\lambda_{sq,4,1,-,j} = \lambda_{sq,4,1,j}, \qquad j = 1, 2 \tag{5.58}$$

$$\lambda_{sq,4,1,-,3} = \lambda_{sq,4,4} = -1 \tag{5.59}$$

$$\lambda_{sq,4,2,+,1} = \lambda_{sq,4,3,2}, \qquad \lambda_{sq,4,2,+,2} = \lambda_{sq,4,3,3} \tag{5.60}$$

$$\lambda_{sq,4,2,-,1} = \lambda_{sq,4,3,1} \tag{5.61}$$

so that

$$\begin{aligned} F(sq[4 \times m, Mb.], q) = &\sum_{j=1}^{5} (\lambda_{sq,4,0,+,j})^m - \sum_{j=1}^{4} (\lambda_{sq,4,0,-,j})^m \\ &+ c^{(1)} \left[\sum_{j=1}^{4} (\lambda_{sq,4,1,+,j})^m - \sum_{j=1}^{3} (\lambda_{sq,4,1,-,j})^m \right] \\ &+ c^{(2)} \left[\sum_{j=1}^{2} (\lambda_{sq,4,2,+,j})^m - (\lambda_{sq,4,2,-,1})^m \right]. \end{aligned} \tag{5.62}$$

For the reader's convenience, in the appendix of the copy of this paper on the math-ph archive, we list specific flow polynomials for some strips of various lattices and widths.

The locus $\mathscr{B}$ for $L_x \to \infty$ is shown in Fig. 3. This locus is again noncompact in the q plane, containing six curves that extend infinitely far away from $q = 0$. The locus separates the q plane into several regions. Three of these regions, R_j, $j = 1, 2, 3$, contain intervals of the real axis, which are $q \geqslant 3$ for R_1, $2 \leqslant q \leqslant 3$ for R_2, and $q < 2$ for R_3. Hence,

$$q_{cf} = 3 \qquad \text{for} \quad \{G\} = sq, 4 \times \infty, cyc./Mb. \tag{5.63}$$

The locus $\mathscr{B}$ again crosses the real axis at $q = 2$ and $q = 3$. In region R_1 the dominant term is the root of maximal magnitude of the cubic equation (5.43), which we denote $\lambda_{sq,4,0,j_{\max}}$, so that

$$fl(sq[4 \times \infty, cyc./Mb.], q) = (\lambda_{sq,4,0,j_{\max}})^{1/3} \qquad \text{for} \quad q \in R_1. \tag{5.64}$$

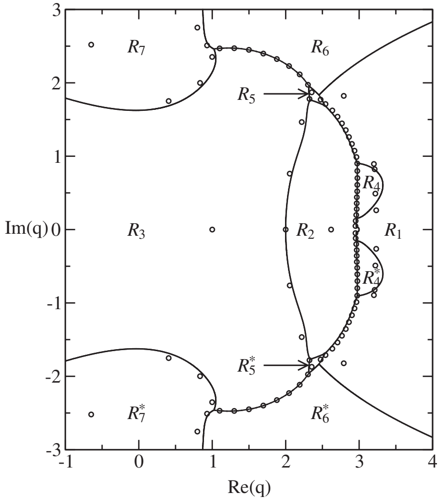

Fig. 3. Singular locus $\mathscr{B}$ in the q plane for $fl(sq, 4 \times \infty, cyc./Mb., q)$ for the $4 \times \infty$ strip of the square lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 4 \times L_x, cyc., q)$ for $L_x = 30$ are shown.

In region R_2, the dominant term is the root of maximal magnitude of the quartic equation (5.48), denoted $\lambda_{sq,4,2,j_{\max}}$, so that

$$|fl(sq[4 \times \infty, cyc./Mb.], q)| = |\lambda_{sq,4,2,j_{\max}}|^{1/3} \quad \text{for} \quad q \in R_2. \tag{5.65}$$

In region R_3, the dominant term is the root of maximal magnitude of the other quartic equation (5.46), so that

$$|fl(sq[4 \times \infty, cyc./Mb.], q)| = |\lambda_{sq,4,1,j_{\max}}|^{1/3} \quad \text{for} \quad q \in R_3. \tag{5.66}$$

In addition to the regions R_j, $j = 1, 2, 3$ that contain intervals of the real axis, there are also four complex-conjugate pairs of regions away from the real axis, R_j, R_j^*, $j = 4, 5, 6, 7$. These can be identified in Fig. 3 as follows: R_4 is a small "bubble" region containing the point $q = 3.1 + 0.6i$; R_6 contains the point $q = 2 + 3i$ and extends to complex infinity; R_7 contains the point $q = 3i$ and extends to complex infinity; and R_5 is a very small region that contains the point $q = 2.4 + 1.8i$.

5.5. On q_{cf} for Cyclic/Möbius Square-Lattice Strips

Aside from the trivial case $L_y = 1$, all of the cyclic/Möbius strips of the square lattice that we have studied have yielded $q_{cf} = 3$. This is the same value as for the infinite square lattice (see Eq. (17.1)). We are led to conjecture that $q_{cf} = 3$ for cyclic/Möbius strips of the square lattice for all $L_y \geqslant 2$. Our present finding and conjecture are related to our previous result that $q_c = 3$ for $\mathscr{B}_W$ for self-dual cyclic strips of the square lattice[63, 75] for all widths considered and our consequent conjecture that this property $q_c = 3$ holds for all cyclic self-dual strips of the square lattice. To see this connection, we recall that the dual graph of the cyclic strip of the square lattice with $L_y \geqslant 2$ is a cyclic strip of this lattice with width $L_y - 1$ augmented by edges joining all of the vertices on the upper and lower sides of the strip to two respective external vertices. The latter strip is rather similar to the cyclic self-dual strips studied in refs. 63 and 75, differing only by the edges joining the lower vertices to the second external point. To the extent that these lower edges do not modify q_c, one would then expect that $q_c = 3$ for the family of dual graphs, which is the equivalent to the conjecture that we make here for q_{cf}.

Our finding that, aside from the trivial case $L_y = 1$, $\mathscr{B}$ for the $L_y \times \infty$ cyclic/Möbius strips of the square lattice crosses the real axis at $q = 2$ for $L_y = 2, 3, 4$ suggests another conjecture, namely that this property holds for all cyclic or Möbius strips with width $L_y \geqslant 2$. It is interesting to relate this to our earlier finding that for this class of cyclic/Möbius strip graphs of the square lattice, $\mathscr{B}_W$ crosses the real axis at $q = 0$ and $q = 2$ for all widths $1 \leqslant L_y \leqslant 5$ considered.[37, 48, 55, 67] One sees that for the widths $L_y \geqslant 2$ considered, $\mathscr{B}_{fl}$ and $\mathscr{B}_W$ both cross the real axis at $q = 2$, while $\mathscr{B}_W$, but not $\mathscr{B}_{fl}$, crosses at $q = 0$ and $\mathscr{B}_{fl}$, but not $\mathscr{B}_W$, crosses at $q = 3$. For the self-dual strips considered in refs. 63 and 75, for which $\mathscr{B}_{fl} = \mathscr{B}_W$, there are crossings at $q = 2, 3$ but not $q = 0$. The actual behavior of $\mathscr{B}_{fl} = \mathscr{B}_W$ for the infinite square lattice would presumably combine these features.

6. APPROACH OF $fl(sq, L_y \times \infty, q)$ TO $fl(sq, q)$

It is of interest to use our exact calculations of the flows per face, $fl(sq, L_y \times \infty, q)$ for the cyclic/Möbius strips of the square lattice to study how the values of this function approach the values for the infinite square lattice as the strip width L_y increases. This is done in Table II for a range of q values. For this table, we define the ratio

$$R_{fl}(sq(L_y), FBC_y, q) = \frac{fl(sq, L_y \times \infty, q)}{fl(sq, q)}. \tag{6.1}$$

We use calculations of $fl(sq, q) = W(sq, q)$ for the infinite square lattice obtained using Monte Carlo methods from refs. 84 and 85. Again, we note

Table II. Comparison of Values of $fl(sq(L_y), \mathrm{FBC}_y, q)$ with $fl(sq, q)$ for $3 \leqslant q \leqslant 10$. For Each Value of q, the Quantities in the Upper Line Are Identified at the Top and the Quantities in the Lower Line Are the Values of $R_{fl}(sq(L_y), \mathrm{FBC}_y, q)$. The FBC_y Is Symbolized as F in the Table

q	$fl(sq(1), F, q)$	$fl(sq(2), F, q)$	$fl(sq(3), F, q)$	$fl(sq(4), F, q)$	$fl(sq, q)$
3	1.4142	1	1	1.1740	1.53960...
	–	0.6495	0.6495	0.7625	
4	1.73205	2	1.7989	1.9520	2.3370(7)
	–	0.8558	0.7697	0.8353	
5	2	3	2.7248	2.8827	3.2510(10)
	–	0.9228	0.8381	0.8867	
6	2.236	4	3.6802	3.8418	4.2003(12)
	–	0.9523	0.8762	0.9146	
7	2.449	5	4.6502	4.8138	5.1669(15)
	–	0.9677	0.9000	0.9317	
8	2.646	6	5.6286	5.7934	6.1431(20)
	–	0.9767	0.9163	0.9431	
9	2.828	7	6.6124	6.7779	7.1254(22)
	–	0.9824	0.9280	0.9512	
10	3	8	7.5998	7.7657	8.1122(25)
	–	0.9862	0.9368	0.9573	

that the case $L_y = 1$ is atypical since the flow polynomial does not exhibit exponential growth with L_x and the number of faces is fixed at two, independent of $L_x = m$, in contrast to all of the strips with $L_y \geqslant 2$. (Hence, for $L_y = 1$ we do not list the R_{fl} values.) For $L_y \geqslant 2$ we find that fl approaches the infinite-width value from below and this approach is not, in general, monotonic. For a given value of $L_y \geqslant 2$, the values of fl are closer to those for the infinite lattice for larger q. In contrast, for cyclic/Möbius strips and for a given value of L_y, W calculated for the infinite strip of width L_y approaches the infinite-width value from above rather than from below.[85]

7. STRIPS OF THE SQUARE LATTICE WITH TORUS AND KLEIN BOTTLE BOUNDARY CONDITIONS

It is of interest to study flows on lattice strip graphs that have doubly periodic boundary conditions. We carry out this study in the present section for the square lattice strips, including both the case of torus and Klein bottle (*Kb.*) topologies.

7.1. $L_y = 2$ Strips of the Square Lattice with Torus and Klein Bottle B.C.

This family involves double edges on each transverse slice of the strip. Although the chromatic polynomial is insensitive to the presence of multiple edges, the flow polynomial is sensitive to this property, as is obvious since in general multiple edges allow more flows satisfying the conservation conditions at each vertex. Using our results in ref. 71 we have

$$\lambda_{sqt,2,0,1} = D_4 = q^2 - 3q + 3 \tag{7.1}$$

where

$$D_m(q) = \frac{P(C_m, q)}{q(q-1)} = \sum_{s=0}^{m-2} (-1)^s \binom{m-1}{s} q^{m-2-s} \tag{7.2}$$

(where the second equality holds for $m \geqslant 2$),

$$\lambda_{sqt,2,1,1} = q^2 - 4q + 5 \tag{7.3}$$

$$\lambda_{sqt,2,2} = 1 \tag{7.4}$$

and

$$\begin{aligned} &F(sq[2 \times L_x = m, torus], q) \\ &\quad = (\lambda_{sqt,2,0,1})^m + c^{(1)}(\lambda_{sqt,2,1,1})^m + c^{(2)} \\ &\quad = (q^2-3q+3)^m + (q-1)(q^2-4q+5)^m + (q^2-3q+1) \end{aligned} \tag{7.5}$$

$$\begin{aligned} &F(sq[2 \times L_x = m, Kb.], q) \\ &\quad = (\lambda_{sqt,2,0,1})^m + c^{(1)}(\lambda_{sqt,2,1,1})^m - 1 \\ &\quad = (q^2-3q+3)^m + (q-1)(q^2-4q+5)^m - 1. \end{aligned} \tag{7.6}$$

We have $C_{F,sq,L_y=2,tor} = (q-1)^2$ for the $2 \times m$ torus strip and $C_{F,sq,L_y=2,Kb.} = q-1$ for the $2 \times m$ Klein bottle strip of the square lattice. Both flow polynomials have $(q-1)$ as a factor, so that the flow number $\phi = 2$ for these families of graphs.

The locus $\mathscr{B}$ is noncompact in the q plane and separates this plane into four regions. This locus is identical to the locus for $\mathscr{B}_W$ in Fig. 5(a) of ref. 43. The region R_1 includes the real segment $q > 2$, while R_2 contain the segment $q < 2$. For this family,

$$q_{cf} = 2 \qquad \text{for} \quad sq,\ 2 \times \infty,\ torus \text{ or } Kb. \tag{7.7}$$

This demonstrates that for a given type of lattice strip and for a given strip width L_y, the imposition of different transverse boundary conditions leads, in general, to a different locus $\mathscr{B}$ and, for the case of periodic longitudinal boundary conditions where $\mathscr{B}$ crosses the real axis, can lead to different values of q_{cf}.

Above and below q_{cf} there are two complex-conjugate phases that extend up and down to triple points, from which c.c. curves extend outward to complex infinity. In R_1, $\lambda_{sqt,2,0,1}$ is dominant, so

$$fl(sq[2\times\infty, torus/Kb.], q) = (q^2-3q+3)^{1/2} \qquad \text{for} \quad q \in R_1 \tag{7.8}$$

where the notation "*torus*/*Kb.*" means that the result holds for the strip with either torus or Klein bottle boundary conditions. In region R_2, $\lambda_{sqt,2,1,1}$ is dominant, so

$$|fl(sq[2\times\infty, torus/Kb.], q)| = |q^2-4q+5|^{1/2} \qquad \text{for} \quad q \in R_2. \tag{7.9}$$

In regions R_3 and R_3^*, $\lambda_{sqt,2,2}$ is dominant, so

$$|fl(sq[2\times\infty, torus/Kb.], q)| = 1 \qquad \text{for} \quad q \in R_3, R_3^*. \tag{7.10}$$

7.2. $L_y = 3$ Strips of the Square Lattice with Torus and Klein Bottle B.C.

Here, for the $L_y = 3$ strips of the square lattice with torus and Klein bottle (*Kb.*) boundary conditions, which we denote generically as $sq3t$ and $sq3kb$, we find

$$N_{F,sq3t,\lambda} = 8 \tag{7.11}$$

$$N_{F,sq3kb,\lambda} = 6. \tag{7.12}$$

These are reductions from the numbers of different terms in the Tutte polynomials for these strips, which are[71] $N_{T,sq3t,\lambda} = 20$ and $N_{T,sq3kb,\lambda} = 12$. These may be compared with the numbers for the chromatic polynomials for these strips[51] $N_{P,sq3t,\lambda} = 8$ and $N_{P,sq3kb,\lambda} = 5$. We have

$$F(sq[3\times m, torus], q) = \sum_{j=1}^{8} c_{sq3t,j}(\lambda_{sq3t,j})^m \tag{7.13}$$

$$F(sq[3\times m, Kb.], q) = \sum_{j=1}^{6} c_{sq3kb,j}(\lambda_{sq3kb,j})^m \tag{7.14}$$

where, in order of increasing degrees of the coefficients (see below)

$$\lambda_{sq3t,1} = q^3 - 6q^2 + 14q - 13 \tag{7.15}$$

$$\lambda_{sq3t,j} = \tfrac{1}{2}\,[-18 + 16q - 6q^2 + q^3 \pm \sqrt{R_{sq3t}}\,] \qquad \text{for} \quad j = 2, 3 \tag{7.16}$$

$$R_{sq3t} = 256 - 440q + 376q^2 - 196q^3 + 64q^4 - 12q^5 + q^6 \tag{7.17}$$

$$\lambda_{sq3t,4} = q - 2 \tag{7.18}$$

$$\lambda_{sq3t,5} = q - 1 \tag{7.19}$$

$$\lambda_{sq3t,6} = q - 4 \tag{7.20}$$

$$\lambda_{sq3t,7} = q - 5 \tag{7.21}$$

$$\lambda_{sq3t,8} = -1. \tag{7.22}$$

In contrast to the situation with cyclic and Möbius strips, there is not a 1–1 correspondence between terms λ and the coefficients that the flow polynomial inherits as a special case of the Tutte polynomial; here, one of the terms appears with two different Tutte coefficients. Specifically, the term $\lambda_{sq3t,4}$ appears with both the coefficient $2(q-1)$ and $q(q-3)$, so that its net coefficient is the sum, $(q+1)(q-2)$. For the coefficients we thus have

$$c_{sq3t,1} = 1 \tag{7.23}$$

$$c_{sq3t,2} = c_{sq3t,3} = q - 1 \tag{7.24}$$

$$c_{sq3t,4} = (q+1)(q-2) \tag{7.25}$$

$$c_{sq3t,5} = \tfrac{1}{2}\, c_{sq3t,6} = \tfrac{1}{2}\,(q-1)(q-2) \tag{7.26}$$

$$c_{sq3t,7} = \tfrac{1}{2}\, q(q-3) \tag{7.27}$$

$$c_{sq3t,8} = q^3 - 6q^2 + 8q - 1. \tag{7.28}$$

The relation of these with the $c^{(d)}$'s was discussed earlier.[58, 67] The sum of coefficients is $C_{F,sq,L_y=3,tor} = (q-1)^3$ for the $3 \times m$ torus strip.

For the Klein bottle strip we have

$$\lambda_{sq3kb,j} = \lambda_{sq3t,j}, \qquad j = 1, 2, 3 \tag{7.29}$$

$$\lambda_{sq3kb,4} = \lambda_{sq3t,5} = q - 1 \tag{7.30}$$

$$\lambda_{sq3kb,5} = \lambda_{sq3t,7} = q - 5 \tag{7.31}$$

$$\lambda_{sq3kb,6} = \lambda_{sq3t,8} = -1. \tag{7.32}$$

Thus the flow polynomial for the $L_y = 3$ strip of the square lattice with torus boundary conditions has two terms, $q-2$ and $q-4$, that are absent from the flow polynomial of this lattice strip with Klein bottle boundary conditions. Evidently, the set of terms for the strip with Klein bottle boundary conditions is a subset of the set of terms for the strip with torus boundary conditions.

The corresponding coefficients are

$$c_{sq3kb,1} = c_{sq3t,1} = 1 \tag{7.33}$$

$$c_{sq3kb,j} = c_{sq3t,j} = q-1, \qquad j = 2, 3 \tag{7.34}$$

$$c_{sq3kb,4} = c_{sq3t,5} = \tfrac{1}{2}(q-1)(q-2) \tag{7.35}$$

$$c_{sq3kb,5} = c_{sq3t,7} = \tfrac{1}{2}q(q-3) \tag{7.36}$$

$$c_{sq3kb,6} = -(q-1). \tag{7.37}$$

The sum of these coefficients is $C_{F,sq,L_y=3,Kb.} = (q-1)^2$. The flow numbers of the sq, $3 \times L_x$ strips with torus and Klein bottle ($Kb.$) boundary conditions are

$$\phi(sq[3 \times L_x, torus/Kb.]) = 2. \tag{7.38}$$

The locus $\mathscr{B}$ is the same for the torus and Klein bottle strips of a given lattice. We have proved this type of equality for $\mathscr{B}_W$ in ref. 58 for chromatic polynomials of torus and Klein bottle strips. The proof for the chromatic polynomials uses the construction of a family of lattice strip graphs which are identical to the torus and Klein bottle strips for even and odd length L_x. The locus $\mathscr{B}$ is the accumulation set of the zeros in the limit $L_x \to \infty$, and this limit can be taken over either even or odd values of L_x, which proves the equality of the loci $\mathscr{B}$ for the respective $L_x \to \infty$ limits of these two types of strips. The same method of proof applies here for the flow polynomials. Note that this result requires that all of the dominant terms are the same for the strips with torus and Klein bottle boundary conditions. This condition is satisfied.

The locus $\mathscr{B}$ for the $L_x \to \infty$ limit of the $L_y = 3$ square strip with torus or Klein bottle boundary conditions is shown in the q and u planes in Figs. 4 and 5. As is evident, $\mathscr{B}$ is noncompact in the q plane and separates this plane into various regions. Three of these regions, R_j, $j = 1, 2, 3$, contain intervals of the real axis, which are $q \geqslant q_{cf}$ for R_1, $2 \leqslant q \leqslant q_{cf}$ for R_2, and $q < 2$ for R_3, where

$$q_{cf} = 2.6120932... \qquad \text{for} \quad sq, 3 \times \infty, torus/Kb. \tag{7.39}$$

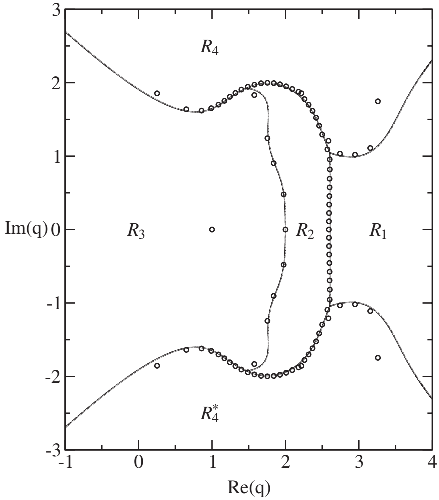

Fig. 4. Singular locus $\mathcal{B}$ in the q plane for $fl(sq, 3\times\infty, torus/Kb., q)$ for the $3\times\infty$ strip of the square lattice with torus or Klein bottle boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 3\times L_x, torus, q)$ for $L_x = 30$ are also shown.

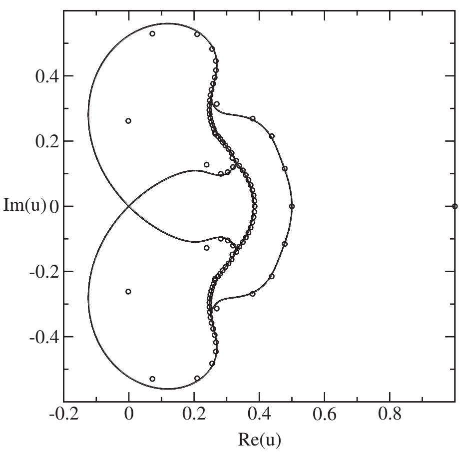

Fig. 5. Singular locus $\mathcal{B}$ in the u plane for $fl(sq, 3\times\infty, torus/Kb., q)$ for the $3\times\infty$ strip of the square lattice with torus or Klein bottle boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 3\times L_x, torus, q)$ for $L_x = 30$ are also shown.

This is the larger of the two real roots of the equation

$$2q^4 - 19q^3 + 71q^2 - 142q + 132 = 0 \tag{7.40}$$

resulting from the condition of degeneracy in magnitude of the leading terms $|\lambda_{sq3t,7}| = |\lambda_{sq3t,2}|$. The fact that q_{cf} increases from the value 2 for $L_y = 2$ to approximately 2.61 for $L_y = 3$ is in accord with the property that as $L_y \to \infty$, $q_{cf} \to 3$ (see Eq. (17.1) later). The behavior exhibited by the strips of the square lattice with torus/Klein bottle boundary conditions is thus that q_{cf} progressively approaches the asymptotic value $q_{cf} = 3$ as L_y increases rather than being equal to this value for each $L_y \geqslant 2$ as we find for the cyclic/Möbius strips of the square lattice.

The dominant term in region R_1 is $\lambda_{sq3t,2}$ so that

$$fl(sq[3 \times \infty, torus/Kb.], q) = (\lambda_{sq3t,2})^{1/3} \quad \text{for} \quad q \in R_1. \tag{7.41}$$

This is interesting since, unlike the flow polynomials for lattice strips that we have studied before, here the dominant term in region R_1 does not have coefficient 1. We recall that the chromatic polynomials for the recursive strips that we have considered always have the property that the dominant term in region R_1 (including the positive real q axis) has coefficient 1.[52] In region R_2, $\lambda_{sq3t,7} = q - 5$ is dominant, so that

$$|fl(sq[3 \times \infty, torus/Kb.], q)| = |q-5|^{1/3} \quad \text{for} \quad q \in R_2. \tag{7.42}$$

In region R_3, $\lambda_{sq3t,3}$ is dominant, so

$$|fl(sq[3 \times \infty, torus/Kb.], q)| = |\lambda_{sq3t,3}|^{1/3} \quad \text{for} \quad q \in R_3. \tag{7.43}$$

The boundary on $\mathscr{B}$ separating regions R_1 and R_2 is the vertical line segment in the q plane given by $Re(q) = q_{cf}$ as given above in Eq. (7.39) and $-1.04 \leqslant Im(q) \lesssim 1.04$. This line segment ends at the complex-conjugate set of triple points at $q = q_{cf} \pm 1.04i$. Going vertically upward and downward away from the real axis along this direction, one passes into two complex-conjugate regions, R_4 and R_4^* in which $\lambda_{sq3t,1}$ is dominant, so that

$$|fl(sq[3 \times \infty, torus/Kb.], q)| = |q^3 - 6q^2 + 14q - 13|^{1/3} \quad \text{for} \quad q \in R_4, R_4^*. \tag{7.44}$$

At the point $u = 0$, four curves on $\mathscr{B}$ intersect. Let us express the terms $\lambda_{sq3t,j}$ in terms of the variable $u = 1/q$ and defining the scaled terms $\lambda_{sq3t,j,u} = u^3 \lambda_{sq3t,j}$. With appropriate choices of branch cuts for the square root, the equation of degeneracy of magnitude of leading terms in the vicinity of $u=0$, when expanded in a small u series, is

$$|1-6u+15u^2+O(u^3)| = |1-6u+14u^2+O(u^3)|. \tag{7.45}$$

Using polar coordinates $u = re^{i\theta}$, this reduces to Eq. (5.41) as $r \to 0$, which shows that the curves approach $u=0$ with the angles $\theta = \pm\pi/4, \pm 3\pi/4$.

8. SELF-DUAL STRIPS OF THE SQUARE LATTICE

In refs. 63 and 75 we calculated chromatic polynomials and Tutte polynomials for families of planar self-dual strips of the square lattice with free and cyclic longitudinal boundary conditions, denoted, respectively, $G_D(L_y \times L_x, cyc.)$ and $G_D(L_y \times L_x, free)$. Applying the relation (3.2), we infer, in particular, that

$$F(G_D, q) = q^{-1} P(G_D, q) \tag{8.1}$$

where G_D refers to any of these graphs. Hence,

$$q_{cf} = q_c = 3 \qquad \text{for } G_D \text{ families.} \tag{8.2}$$

Similarly, the loci $\mathscr{B}$ are the same for the $L_x \to \infty$ limits of the flow and chromatic polynomials for these families of graphs. A particular property is that, in contrast to the loci $\mathscr{B}$ for the other families studied here, these are compact in the q plane.

9. STRIPS OF THE SQUARE LATTICE WITH FREE BOUNDARY CONDITIONS

It is of interest to compare the flow polynomials for strips with periodic or twisted periodic longitudinal boundary conditions to those with free boundary conditions. These flow polynomials may be calculated directly by iterative application of the deletion-contraction property or via the chromatic polynomials of the dual graphs, or as special cases of Tutte polynomials. We list some relevant results. Clearly for a tree graph, the flow polynomial vanishes. For strips with free longitudinal boundary conditions, we shall use a different labelling convention, viz., $m = L_x - 1$, than the convention $m = L_x$ used for strips with periodic longitudinal boundary conditions; in both cases, m refers to the length of the respective strips in terms of edges with L_x being the length in terms of vertices. For $m \geqslant 1$ we have

$$F(sq[L_y=2, L_x=m+1, free], q) = (q-1)(q-2)^{m-1} \tag{9.1}$$

$$F(sq[L_y=3, L_x=m+1, free], q) = (q-1)(q-2)(q^2-5q+7)^{m-1}. \tag{9.2}$$

Thus, $N_{F, sq, L_y, \lambda} = 1$ for $L_y = 2, 3$. For both of these families of strip graphs, the continuous accumulation set of zeros is the empty set since the zeros are discrete.

It is convenient to use a generating function to give the results for the cases $L_y \geqslant 4$. This is similar to the generating functions that were utilized for chromatic polynomials in refs. 39 and 41 and our subsequent works. For the strip of type G, length $L_x = m+1$, we write

$$\Gamma(G, q, z) = \sum_{m=0}^{\infty} F(G_m, q)\, z^m \tag{9.3}$$

where

$$\Gamma(G, q, z) = \frac{\mathcal{N}(G, q, z)}{\mathcal{D}(G, q, z)} \tag{9.4}$$

with

$$\mathcal{N}(G, q, z) = \sum_{j=0}^{d_{\mathcal{N}}} A_{G,j}(q)\, z^j \tag{9.5}$$

and

$$\mathcal{D}(G, q, z) = 1 + \sum_{j=1}^{d_{\mathcal{D}}} b_{G,j}(q)\, z^j \tag{9.6}$$

$$\mathcal{N}(sq[L_y=4, free], q, z) = (q-1)(q-2)\, z[(q-2)-(q-1)(q-3)\, z] \tag{9.7}$$

$$\begin{aligned}\mathcal{D}(sq[L_y=4, free], q, z) = 1 &- (q^3-8q^2+24q-26)\, z \\ &- (q^5-12q^4+59q^3-149q^2+193q-101)\, z^2 \\ &+ (q-2)^5\, (q-3)\, z^3.\end{aligned} \tag{9.8}$$

Thus, $N_{F, sq, L_y=4, free, \lambda} = 3$.

The locus $\mathcal{B}$ for the $L_x \to \infty$ limit, shown in Fig. 6, is compact and consists of complex-conjugate pairs of arcs together with a self-conjugate

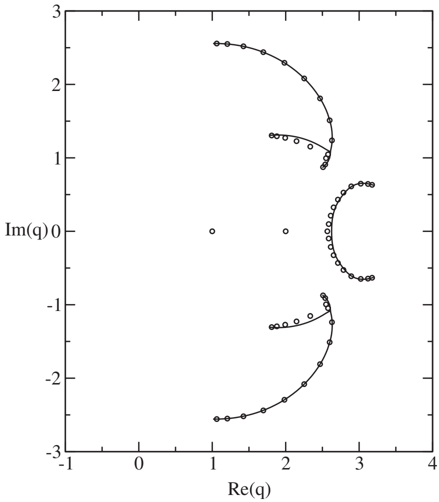

Fig. 6. Singular locus $\mathscr{B}$ in the q plane for $fl(sq, 4\times\infty, free, q)$ for the $4\times\infty$ strip of the square lattice with free boundary conditions. For comparison, zeros of the flow polynomial $F(sq, 4\times L_x, free, q)$ for $L_x=20$ are also shown.

arc that crosses the real axis at $q \simeq 2.6252$. This is similar to what we found for the loci $\mathscr{B}_W$ for strips with free boundary conditions in earlier work.[39, 41] Although $\mathscr{B}$ crosses the real axis for this strip, it is not guaranteed to cross the real axis when one uses free longitudinal boundary conditions, as will be illustrated by an explicit example later.

We have also calculated the generating function for $F(sq[L_y=5,\ L_x, free], q)$ and find that $N_{F,\,sq,\,L_y=5,\,free,\,\lambda}=4$, with

$$A_{sq,\,5,\,free,\,0}=0 \tag{9.9}$$

$$A_{sq,\,5,\,free,\,1}=(q-1)(q-2)^3 \tag{9.10}$$

$$A_{sq,\,5,\,free,\,2}=-(q-1)(q-2)(q^5-8q^4+22q^3-20q^2-5q+9) \tag{9.11}$$

$$A_{sq,\,5,\,free,\,3}=(q-1)(q-2)(2q^7-29q^6+178q^5-598q^4+1186q^3 \\ -1387q^2+884q-237) \tag{9.12}$$

$$A_{sq,\,5,\,free,\,4}=-(q-1)(q-2)^4\,(q^6-14q^5+79q^4-229q^3+359q^2-288q+91) \tag{9.13}$$

$$b_{sq,\,5,\,free,\,1}=-q^4+10q^3-44q^2+97q-88 \tag{9.14}$$

$$b_{sq,\,5,\,free,\,2}=-q^7+19q^6-156q^5+726q^4-2085q^3+3711q^2-3790q+1708 \tag{9.15}$$

$$b_{sq,\,5,\,free,\,3} = (q-3)(2q^8 - 39q^7 + 337q^6 - 1685q^5 + 5335q^4 - 10959q^3 + 14264q^2 - 10753q + 3595) \tag{9.16}$$

$$b_{sq,\,5,\,free,\,4} = -(q-2)^2\,(q^9 - 23q^8 + 235q^7 - 1401q^6 + 5376q^5 - 13785q^4 + 23647q^3 - 26198q^2 + 17033q - 4964). \tag{9.17}$$

The locus $\mathscr{B}$ is comprised of arcs and is similar to, although more complicated than, that for the $L_y = 4$ strip. The deletion-contraction property (2.1) can also be used in an interative manner to calculate flow polynomials for strips with greater widths.

10. STRIPS OF THE SQUARE LATTICE WITH CYLINDRICAL BOUNDARY CONDITIONS

For strips of the square lattice with cylindrical boundary conditions we calculate, for $m \geqslant 1$,

$$\begin{aligned} F(sq[L_y = 2, L_x = m+1, cyl.], q) &= (q-1)(q-2)^2\,(D_4)^{m-1} \\ &= (q-1)(q-2)^2\,(q^2 - 3q + 3)^{m-1} \end{aligned} \tag{10.1}$$

$$\begin{aligned} F(sq[L_y = 3, L_x = m+1, cyl.], q) &= (q-1)(q-2)(q-3)^2\,(q^3 - 6q^2 + 14q - 13)^{m-1}. \end{aligned} \tag{10.2}$$

For both of these families of strip graphs, the continuous accumulation set of zeros is empty since the zeros are discrete.

The coefficients of the generating function for $F(sq[L_y = 4, L_x, cyl.], q)$ are

$$A_{sq,\,4,\,cyl,\,0} = q-1 \tag{10.3}$$

$$A_{sq,\,4,\,cyl,\,1} = -(q-1)(3q^3 - 18q^2 + 35q - 18) \tag{10.4}$$

$$A_{sq,\,4,\,cyl,\,2} = (q-1)^2\,(q-3)(q^3 - 7q^2 + 13q - 9) \tag{10.5}$$

$$b_{sq,\,4,\,cyl,\,1} = -q^4 + 8q^3 - 29q^2 + 55q - 46 \tag{10.6}$$

$$b_{sq,\,4,\,cyl,\,2} = q^6 - 12q^5 + 61q^4 - 169q^3 + 269q^2 - 231q + 85. \tag{10.7}$$

Thus, $N_{F,\,sq,\,L_y=4,\,cyl,\,\lambda} = 2$, and the terms are

$$\lambda_{F,\,sq,\,L_y=4,\,cyl,\,j} = \tfrac{1}{2}\,[q^4 - 8q^3 + 29q^2 - 55q + 46 \pm \sqrt{R_{sq4c}}] \tag{10.8}$$

where

$$R_{sq4c} = q^8 - 16q^7 + 118q^6 - 526q^5 + 1569q^4 - 3250q^3 + 4617q^2 - 4136q + 1776. \tag{10.9}$$

The denominator $\mathcal{D}$ of the generating function is the same as that for the generating function for chromatic polynomials of the $4 \times L_x$ strip of the square lattice with cylindrical boundary conditions, and consequently, in the $L_x \to \infty$ limit, the locus $\mathcal{B}$ is also the same as the locus $\mathcal{B}_W$ for the $4 \times \infty$ square-lattice strip with cylindrical boundary conditions, shown as Fig. 3(a) in ref. 41. This locus is compact and consists of a complex-conjugate pair of arcs together with a self-conjugate arc that crosses the real axis (at $q \simeq 2.30$) and a very short line segment emanating outward from this crossing point on the real axis. The value of q_{cf}, which is given by the right-hand end of the very short line segment, occurs at $q \simeq 2.35$. The locus does not separate the q plane into different regions.

The coefficients of the generating function for $F(sq[L_y = 5, L_x, cyl.], q)$ are

$$A_{sq,\,5,\,cyl,\,0} = q - 1 \tag{10.10}$$

$$A_{sq,\,5,\,cyl,\,1} = -(q-1)(4q^4 - 35q^3 + 115q^2 - 157q + 62) \tag{10.11}$$

$$A_{sq,\,5,\,cyl,\,2} = -(q-1)^2\,(q^6 - 7q^5 - q^4 + 133q^3 - 458q^2 + 645q - 348) \tag{10.12}$$

$$b_{sq,\,5,\,cyl,\,1} = -q^5 + 10q^4 - 46q^3 + 124q^2 - 198q + 148 \tag{10.13}$$

$$\begin{aligned} b_{sq,\,5,\,cyl,\,2} &= q^8 - 19q^7 + 159q^6 - 767q^5 + 2339q^4 \\ &\quad - 4627q^3 + 5800q^2 - 4212q + 1362. \end{aligned} \tag{10.14}$$

Thus, $N_{sq,\,L_y = 5,\,cyl,\,\lambda} = 2$. As was the case with $L_y = 4$, the denominator $\mathcal{D}$ of the generating function is the same as that for the generating function for chromatic polynomials of the $5 \times L_x$ strip of the square lattice with cylindrical boundary conditions, and consequently, in the $L_x \to \infty$ limit, the locus $\mathcal{B}$ is also the same as the locus $\mathcal{B}_W$ for the $5 \times \infty$ square-lattice strip with cylindrical boundary conditions, shown as Fig. 2 of ref. 55. As before, $\mathcal{B}$ is compact and consists of five arcs, four of which form two complex-conjugate pairs and one of which is self-conjugate. The self-conjugate arc crosses the real axis at $q_{cf} \simeq 2.69168$. The endpoints of the arcs occur at the ten zeros of the polynomial $b_{sq,\,5,\,cyl,\,1}^2 - 4b_{sq,\,5,\,cyl,\,2}$, which square roots that occur in $\lambda_{sq,\,L_y = 5,\,cyl,\,j}$ have branch point singularities. The arcs comprising $\mathcal{B}$ do not separate the q plane into different regions. As mentioned above, the deletion-contraction property (2.1) can also be used in an interative manner to calculate flow polynomials for strips with greater widths.

11. $L_y = 2, 3$ CYCLIC STRIP OF THE HONEYCOMB LATTICE

The $L_y = 2$ cyclic or Möbius strip of the honeycomb lattice can be constructed by starting with the $L_y = 2$ cyclic or Möbius strip of the square lattice and inserting degree-2 vertices on each horizontal (longitudinal) edge. Because the insertion of degree-2 vertices does not affect the flow polynomial, it follows that, for $\mathrm{BC}_x = \mathrm{FBC}_x, \mathrm{PBC}_x, \mathrm{TPBC}_x$, and $L_y = 2$, we have

$$F(hc[2 \times L_x, \mathrm{FBC}_y, \mathrm{BC}_x], q) = F(sq[2 \times L_x, \mathrm{FBC}_y, \mathrm{BC}_x], q) \qquad (11.1)$$

and

$$fl(hc[2 \times \infty, cyc./Mb.], q) = fl(sq[2 \times \infty, cyc./Mb.], q) \qquad (11.2)$$

where the expressions for $F(sq, 2 \times L_x, \mathrm{FBC}_y, \mathrm{BC}_x, q)$ with the various longitudinal boundary conditions were given above.

For $L_y = 3$ we calculate

$$F(hc[3 \times m, cyc.], q)$$
$$= (\lambda_{hc,3,0,1})^m + c^{(1)} \sum_{j=1}^{3} (\lambda_{hc,3,1,j})^m + c^{(2)} \sum_{j=1}^{2} (\lambda_{hc,3,2,j})^m + c^{(3)} \qquad (11.3)$$

where

$$\lambda_{hc,3,0,1} = (q-3)^2 \qquad (11.4)$$

$$\lambda_{hc,3,2,j} = \tfrac{1}{2}\,[7 - 2q \pm \sqrt{13-4q}] \qquad \text{for} \quad j = 1, 2 \qquad (11.5)$$

$$\lambda_{hc,3,3} = 1 \qquad (11.6)$$

and the $\lambda_{hc,3,1,j}$, $j = 1, 2, 3$ are the roots of the equation

$$\xi^3 - (q-3)(q-5)\,\xi^2 - (q-3)^2\,(2q-5)\,\xi - (q-2)^2\,(q-3)^2 = 0. \qquad (11.7)$$

Hence,

$$\phi(hc[3 \times m, cyc.]) = 4. \qquad (11.8)$$

The locus $\mathscr{B}$ for the $L_x \to \infty$ limit of the cyclic or Möbius $L_y = 3$ strips of the honeycomb lattice is shown in the q plane in Fig. 7 and in the u plane in Fig. 8. We have

$$q_{cf} = \frac{5+\sqrt{5}}{2} = 3.61803... \qquad \text{for} \quad hc, 3 \times \infty, cyc./Mb. \qquad (11.9)$$

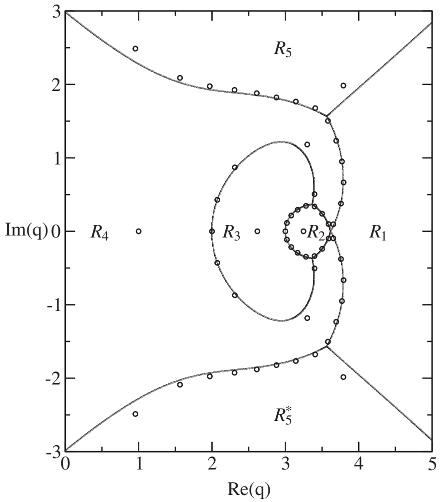

Fig. 7. Singular locus $\mathscr{B}$ in the q plane for $fl(hc, 3\times\infty, cyc./Mb., q)$ for the $3\times\infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(hc, 3\times L_x, cyc., q)$ for $L_x = 30$ are also shown.

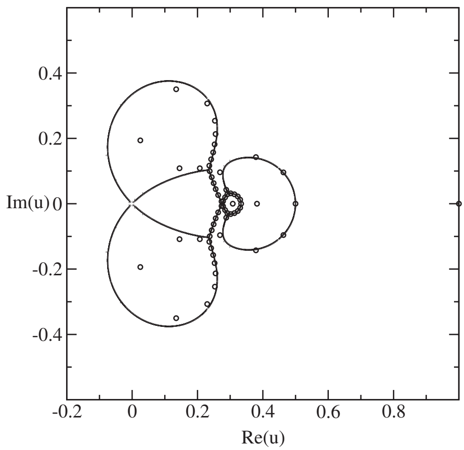

Fig. 8. Singular locus $\mathscr{B}$ in the u plane for $fl(hc, 3\times\infty, cyc./Mb., q)$ for the $3\times\infty$ strip of the honeycomb lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(hc, 3\times L_x, cyc., q)$ for $L_x = 30$, expressed in terms of u, are also shown.

This is already within 10% of the asymptotic value $q_{cf}=4$ for the 2D honeycomb lattice (see Eq. (17.2) later). The locus $\mathscr{B}$ is again noncompact in the q plane, passing through the origin of the u plane. The locus separates the q plane into several regions, which contain intervals of the real axis: R_j, $j=1, 2, 3, 4$, which include the respective intervals (1) $q \geqslant q_{cf}$, (2) $3 \leqslant q \leqslant q_{cf}$, (3) $2 \leqslant q \leqslant 3$, and (4) $q \leqslant 2$. In region R_1, the dominant term is the root of the cubic equation (11.7) with maximal magnitude, which we denote as $\lambda_{hc, 3, 1, 1}$, so that

$$fl(hc[3\times\infty, cyc./Mb.], q) = (\lambda_{hc, 3, 1, 1})^{1/2} \qquad \text{for} \quad q \in R_1. \tag{11.10}$$

In region R_2, $\lambda_{hc, 3, 3}=1$ is dominant, so

$$|fl(hc[3\times\infty, cyc./Mb.], q)| = 1 \qquad \text{for} \quad q \in R_2. \tag{11.11}$$

In region R_3, $\lambda_{hc, 3, 2, 1}$ is dominant, so

$$|fl(hc[3\times\infty, cyc./Mb.], q)| = |\tfrac{1}{2}(7-2q+\sqrt{13-4q})|^{1/2} \qquad \text{for} \quad q \in R_3. \tag{11.12}$$

In region R_4 the maximal root of the cubic equation (11.7) is dominant, with a result analogous to (11.10) for fl. In regions R_5 and R_5^* the dominant λ is $\lambda_{hc, 3, 0, 1}$, so that

$$|fl(hc[3\times\infty, cyc./Mb.], q)| = |q-3| \qquad \text{for} \quad q \in R_5, R_5^*. \tag{11.13}$$

The curves cross the origin of the u plane with the angles $\pm\pi/4$ and $3\pi/4$.

12. $L_y=2$ CYCLIC STRIP OF THE TRIANGULAR LATTICE

Next, we consider strips of the triangular lattice. For the $L_y=2$ cyclic strip we calculate

$$\lambda_{tri, 2, 0, 1} = (q-2)^2 \tag{12.1}$$

$$\lambda_{tri, 2, 1, j} = \tfrac{1}{2}[6-4q+q^2 \pm (q-2)\sqrt{8-4q+q^2}] \tag{12.2}$$

where $j=1, 2$ for the $\pm$ sign before the square root, and

$$\lambda_{tri, 2, 2} = 1. \tag{12.3}$$

Then

$$F(tri[2\times m, cyc.], q) = (q-2)^{2m} + c^{(1)}[(\lambda_{tri, 2, 1, 1})^m + (\lambda_{tri, 2, 1, 1})^m] + c^{(2)}. \tag{12.4}$$

Thus, $N_{F,\,tri,\,L_y=2,\,cyc.,\,\lambda}=4$, and for the cyclic strip of the triangular lattice of width $L_y=2$,

$$n_F(tri,2,0)=1, \qquad n_F(tri,2,1)=2, \qquad n_F(tri,2,2)=1 \tag{12.5}$$

with $n_F(tri,2,d)=0$ for $d \geqslant 3$. For this strip the sum of coefficients is

$$C_{F,\,tri,\,L_y=2}=q(q-1) \qquad \text{for} \quad tri,\, 2\times L_x,\, cyc. \tag{12.6}$$

Using our exact solution, we observe that $F(tri[2\times m, cyc.], q)$ has only the common factor $(q-1)$ and hence

$$\phi(tri[2\times m, cyc.])=2. \tag{12.7}$$

Further, we find that

$$\phi(tri[2\times m, Mb.])=3. \tag{12.8}$$

The locus $\mathscr{B}$ for the cyclic and Möbius $L_y=2$ strips of the triangular lattice is shown in the q plane in Fig. 9 and in the u plane in Fig. 10. This locus is noncompact in the q plane and divides this plane into four regions, R_1, R_2, and the complex-conjugate regions R_3 and R_3^*. Regions R_1 and R_2

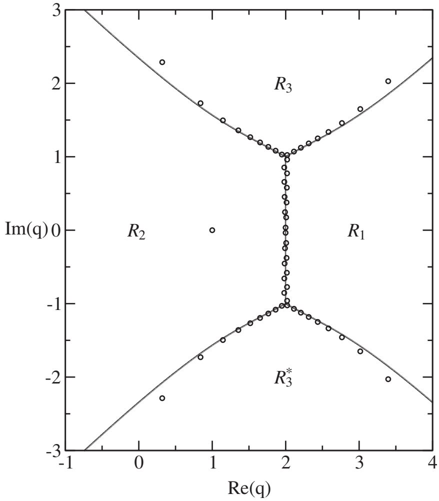

Fig. 9. Singular locus $\mathscr{B}$ in the q plane for $fl(tri, 2\times\infty, cyc./Mb., q)$ for the $2\times\infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 2\times L_x, cyc., q)$ for $L_x=30$ (so that this polynomial has degree 61) are also shown.

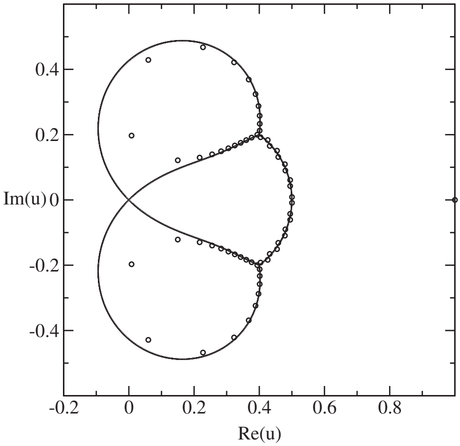

Fig. 10. Singular locus $\mathcal{B}$ in the u plane for $fl(tri, 2\times\infty, cyc./Mb., q)$ for the $2\times\infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 2\times L_x, cyc., q)$ for $L_x=30$, expressed in terms of u, are also shown.

contain the respective real intervals $q>2$ and $q<2$ and are contiguous along a vertical line segment extending from $q=2+i$ to $2-i$. Triple points on $\mathcal{B}$ occur at these endpoints $q=2\pm i$. Evidently,

$$q_{cf}=2 \qquad \text{for} \quad tri,\ 2\times\infty,\ cyc./Mb. \tag{12.9}$$

Along the vertical line segment from $2+i$ to $2-i$, $|\lambda_{tri,2,1,1}|=|\lambda_{tri,2,1,2}|$ $=\lambda_{tri,2,2}=1$. Regions R_3 and R_3^* extend upward and downward, respectively, from the complex-conjugate triple points at $q=2+i$ and $q=2-i$. In region R_1, $\lambda_{tri,2,1,1}$ is dominant so that

$$fl(tri[2\times\infty, cyc.], q)=f(tri[2\times\infty, Mb.], q)=(\lambda_{tri,2,1,1})^{1/2} \qquad \text{for} \quad q\in R_1. \tag{12.10}$$

In region R_2, $\lambda_{tri,2,1,2}$ is dominant, so that

$$|fl(tri[2\times\infty, cyc./Mb.], q)|=|\lambda_{tri,2,1,2}|^{1/2} \qquad \text{for} \quad q\in R_2. \tag{12.11}$$

In regions R_3 and R_3^*, $\lambda_{tri,2,0,1}$ is dominant, so that

$$|fl(tri[2\times\infty, cyc./Mb.], q)|=|q-2| \qquad \text{for} \quad q\in R_3,\ R_3^*. \tag{12.12}$$

Some special values include $fl(2)=1$, $|fl(1)|=[(3+\sqrt{5})/2]^{1/2}$, and $|fl(0)|$ $=[3+2\sqrt{2}]^{1/2}$.

To show that $\mathscr{B}$ passes through $u=0$, we calculate the corresponding scaled terms $\lambda_{tri,2,d,j,u}=\lambda_{tri,2,d,j}/q^2$ with $u=1/q$ and see that the degeneracy condition of dominant $\lambda_{tri,2,d,j,u}$'s has a solution at $u=0$. The dominant $\lambda_{tri,2,d,j,u}$'s in the vicinity of $u=0$ are

$$\lambda_{tri,2,0,1,u}=(1-2u)^2 \tag{12.13}$$

$$\lambda_{tri,2,1,1,u}=\tfrac{1}{2}\left[1-4u+6u^2+(1-2u)\sqrt{1-4u+8u^2}\right]. \tag{12.14}$$

For $u\to 0$, the latter term has the Taylor series expansion

$$\lambda_{tri,2,1,1,u}=1-4u+6u^2-u^4+O(u^5). \tag{12.15}$$

Thus, in the same way as before, introducing polar coordinates and expanding the equation of the degeneracy of magnitudes of leading terms in the vicinity of $u=0$ yields the condition $r^2\cos 2\theta=0$, thereby showing that four branches of $\mathscr{B}$ approach the origin of the u plane at the angles given by (5.42), i.e., $\theta=\pm\pi/4$ and $\theta=\pm 3\pi/4$.

13. $L_y=3$ CYCLIC STRIP OF THE TRIANGULAR LATTICE

For the $L_y=3$ cyclic strip of the triangular lattice we find the following results:

$$\begin{aligned}\lambda_{tri,3,0,j}=\tfrac{1}{2}\big[&q^4-7q^3+21q^2-33q+23\\&\pm(q^8-14q^7+91q^6-360q^5+949q^4-1708q^3\\&+2047q^2-1486q+497)^{1/2}\big] \qquad j=1,2\end{aligned} \tag{13.1}$$

$$\lambda_{tri,3,1,j}=q^2-4q+5. \tag{13.2}$$

The $\lambda_{tri,3,1,j}$ for $j=2,3,4$ are the roots of the equation

$$\begin{aligned}\xi^3-(q^4-7q^3+22q^2-37q+30)\,\xi^2&\\+(q^6-11q^5+52q^4-134q^3+200q^2-168q+69)\,\xi&\\-(2q^4-15q^3+42q^2-51q+24)=0&\end{aligned} \tag{13.3}$$

$$\lambda_{tri,3,2,1}=q^2-3q+3 \tag{13.4}$$

$$\lambda_{tri,3,2,j}=\tfrac{1}{2}\left[q^2-5q+9\pm(q^4-10q^3+43q^2-90q+73)^{1/2}\right], \qquad j=2,3 \tag{13.5}$$

$$\lambda_{tri,3,3}=1. \tag{13.6}$$

Hence, $N_{F,\,tri,\,L_y=3,\,cyc.,\,\lambda}=10$, $n_F(tri,3,0)=2$, $n_F(tri,3,1)=4$, $n_F(tri,3,2)=3$, $n_F(tri,3,3)=1$, and

$$\begin{aligned} &F(tri[3\times m, cyc.], q) \\ &\quad =\sum_{j=1}^{2}(\lambda_{tri,3,0,j})^m+c^{(1)}\sum_{j=1}^{4}(\lambda_{tri,3,1,j})^m+c^{(2)}\sum_{j=1}^{3}(\lambda_{tri,3,2,j})^m+c^{(3)}. \end{aligned} \tag{13.7}$$

Using our exact result (13.7) we observe that $F(tri[3\times m, cyc.], q)$ has only the common factor $(q-1)$, so that

$$\phi(tri[3\times m, cyc.])=2. \tag{13.8}$$

This can also be derived as a corollary of the basic theorem discussed in Section 3.2, that a bridgeless graph admits a 2-flow if and only if all of its vertex degrees are even; here all of the vertices of the $tri[3\times m, cyc.]$ are even.

For the cyclic strip of the triangular lattice of width $L_y=3$, the sum of coefficients is

$$C_{F,\,tri,\,L_y=3}=q(q-1)^2 \qquad \text{for} \quad tri,\ 3\times L_x,\ cyc. \tag{13.9}$$

The locus $\mathscr{B}$ for the cyclic and Möbius $L_y=3$ strips of the triangular lattice is shown in the q plane in Fig. 11. This locus is noncompact in the q plane, containing eight curves that extend infinitely far away from $q=0$. The locus separates the q plane into several regions. Three of these regions, R_j, $j=1,2,3$, contain interval of the real axis, which are $q\geqslant q_{cf}$ for R_1, and $2\leqslant q\leqslant q_{cf}$ for R_2, and $q<2$ for R_3, where

$$q_{cf}=2.213548 \qquad \text{for} \quad tri,\ 3\times\infty,\ cyc./Mb. \tag{13.10}$$

This is within about 15% of the asymptotic value $q_{cf}\simeq 2.618$ for the 2D triangular lattice (see Eq. (17.3) later). In regions R_1 and R_3, the dominant term is the root of maximal magnitude of the cubic equation (13.3), which we denote $\lambda_{tri,3,1,j_{\max}}$, so that

$$fl(tri[3\times\infty, cyc./Mb.], q)=(\lambda_{tri,3,1,j_{\max}})^{1/4} \qquad \text{for} \quad q\in R_1. \tag{13.11}$$

In region R_2, $\lambda_{tri,3,2,2}$ is dominant, so that

$$|fl(tri[3\times\infty, cyc./Mb.], q)|=|\lambda_{tri,3,2,2}|^{1/4} \qquad \text{for} \quad q\in R_2. \tag{13.12}$$

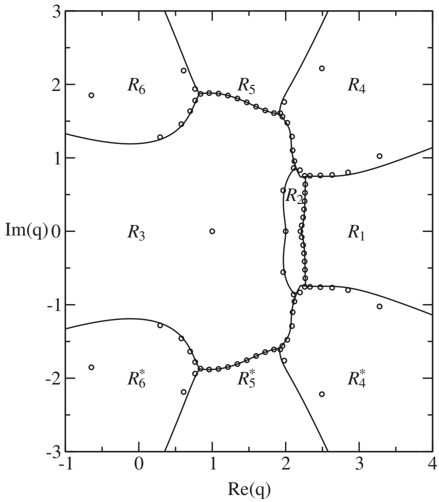

Fig. 11. Singular locus $\mathscr{B}$ in the q plane for $fl(tri, 3 \times \infty, cyc./Mb., q)$ for the $3 \times \infty$ strip of the triangular lattice with cyclic or Möbius boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 3 \times L_x, cyc., q)$ for $L_x = 20$ (so that this polynomial has degree 81) are also shown.

In region R_3,

$$|fl(tri[3 \times \infty, cyc./Mb.], q)| = |\lambda_{tri,3,1,j_{\max}}|^{1/4} \qquad \text{for} \quad q \in R_3. \qquad (13.13)$$

In addition to the regions R_j, $j = 1, 2, 3$ that contain intervals of the real axis, there are also three complex-conjugate pairs of regions away from the real axis, R_j, R_j^*, $j = 4, 5, 6$. These can be identified in Fig. 11 as follows: R_4 contains the point $q = 4 + 3i$ and extends to complex infinity; R_5 contains the point $q = 1.5 + 3i$ and extends to complex infinity; and R_6 contains the point $q = -1 + 3i$ and extends to complex infinity.

14. $L_y = 2$ STRIP OF THE TRIANGULAR LATTICE WITH TOROIDAL AND KLEIN BOTTLE BOUNDARY CONDITIONS

For the $L_y = 2$ strips of the triangular lattice with torus and Klein bottle boundary conditions, we find that

$$\lambda_{tri2t,j} = \tfrac{1}{2}\,[11 - 19q + 15q^2 - 6q^3 + q^4 \pm (129 - 446q + 727q^2 - 722q^3 + 479q^4 - 218q^5 + 66q^6 - 12q^7 + q^8)^{1/2}] \qquad j = 1, 2 \qquad (14.1)$$

$$\lambda_{tri2t,j} = \tfrac{1}{2}\,[14 - 20q + 15q^2 - 6q^3 + q^4 \pm (212 - 600q + 852q^2 - 776q^3 + 493q^4 - 220q^5 + 66q^6 - 12q^7 + q^8)^{1/2}] \qquad j = 3, 4 \qquad (14.2)$$

and

$$\lambda_{tri2t,5} = 2. \tag{14.3}$$

The corresponding coefficients are

$$c_{tri2t,j} = 1 \qquad j = 1, 2 \tag{14.4}$$

$$c_{tri2t,j} = q-1 \qquad j = 3, 4 \tag{14.5}$$

and

$$c_{tri2t,5} = \tfrac{1}{2}\, q(q-3). \tag{14.6}$$

It follows that

$$F(tri[2\times m,\ torus],\ q) = (\lambda_{tri2t,1})^m + (\lambda_{tri2t,2})^m + (q-1)[(\lambda_{tri2t,3})^m + (\lambda_{tri2t,4})^m] + \tfrac{1}{2}\, q(q-3)\, 2^m. \tag{14.7}$$

The locus $\mathscr{B}$ for the torus and Klein bottle $L_y = 2$ strips of the triangular lattice is shown in the q plane in Fig. 12 and in the u plane in Fig. 13. This locus is noncompact in the q plane, and divides this plane into various

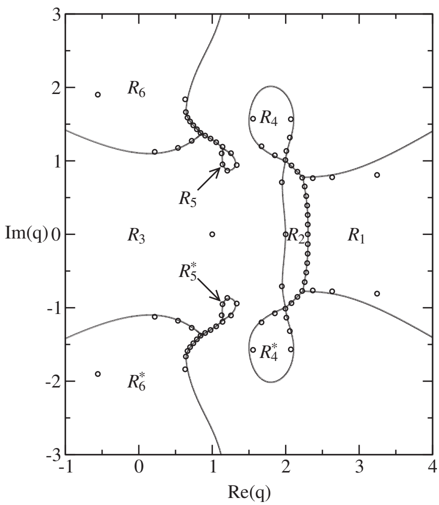

Fig. 12. Singular locus $\mathscr{B}$ in the q plane for $fl(tri, 2\times\infty, torus/Kb., q)$ for the $2\times\infty$ strip of the triangular lattice with torus or Klein bottle boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 2\times L_x, torus, q)$ for $L_x = 20$ are also shown.

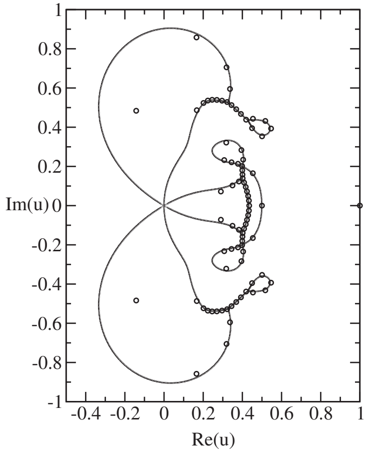

Fig. 13. Singular locus $\mathscr{B}$ in the u plane for $fl(tri, 2\times\infty, torus/Kb., q)$ for the $2\times\infty$ strip of the triangular lattice with torus or Klein bottle boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 2\times L_x, torus, q)$ for $L_x = 20$, expressed in terms of u, are also shown.

regions. Three of these regions, R_j, $j = 1, 2, 3$, contain intervals of the real axis, which are $q \geqslant q_{cf}$ for R_1, $2 \leqslant q \leqslant q_{cf}$ for R_2, and $q < 2$ for R_3, where

$$q_{cf} = 2.307144568... \qquad \text{for} \quad tri, 2\times\infty, torus/Kb. \tag{14.8}$$

The dominant terms in regions R_j, $j = 1, 2, 3$ are $\lambda_{tri2t,1}$, $\lambda_{tri2t,5}$, and $\lambda_{tri2t,3}$ so that

$$fl(tri[2\times\infty, torus/Kb.], q) = (\lambda_{tri2t,1})^{1/4} \qquad \text{for} \quad q \in R_1 \tag{14.9}$$

$$|fl(tri[2\times\infty, torus/Kb.], q)| = 2^{1/4} \qquad \text{for} \quad q \in R_2 \tag{14.10}$$

$$|fl(tri[2\times\infty, torus/Kb.], q)| = |\lambda_{tri2t,3}|^{1/4} \qquad \text{for} \quad q \in R_3. \tag{14.11}$$

There are also three complex-conjugate pairs of regions, R_j, R_j^*, $j = 4, 5, 6$. These can be identified in Fig. 12 as follows: R_4 is a "bubble" region centered at the point $q = 1.8 + 1.5i$; R_5 is a small "bubble" region centered at the point $q = 1.2 + i$; and R_6 contains the point $q = 2i$ and extends to complex infinity. The dominant terms in regions R_j, $j = 4, 5, 6$ are $\lambda_{tri2t,1}$, $\lambda_{tri2t,5}$ and $\lambda_{tri2t,1}$.

To show that $\mathscr{B}$ passes through $u = 0$, we calculate the corresponding scaled terms $\lambda_{tri2t,j,u} = \lambda_{tri2t,j}/q^4$ with $u = 1/q$ and see that the degeneracy condition of dominant $\lambda_{tri2t,j,u}$'s has a solution at $u = 0$. The dominant terms in the vicinity of $u=0$ are $\lambda_{tri2t,1,u}$ and $\lambda_{tri2t,3,u}$. For $u\to 0$, these terms have the Taylor series expansion

$$\lambda_{tri2t,1,u} = 1-6u+15u^2-19u^3+O(u^4) \tag{14.12}$$

$$\lambda_{tri2t,3,u} = 1-6u+15u^2-20u^3+O(u^4). \tag{14.13}$$

Hence, introducing polar coordinates $u=re^{i\theta}$, the condition of degeneracy of magnitudes in the vicinity of $u=0$ yields the condition

$$r^3\cos\theta(3-4\cos^2\theta)=0 \qquad \text{as} \quad r\to 0. \tag{14.14}$$

This proves that four branches of $\mathscr{B}$ approach the origin of the u plane at the angles

$$\theta=\frac{(2k+1)\,\pi}{6}, \qquad \text{for} \quad 1\leqslant k\leqslant 6 \tag{14.15}$$

i.e., at $\theta=\pm\pi/6$, $\theta=\pm\pi/2$ and $\theta=\pm 5\pi/6$.

15. STRIPS OF THE TRIANGULAR LATTICE WITH FREE BOUNDARY CONDITIONS

In this section, we give the flow polynomials of the triangular lattice with free boundary conditions. First, we have

$$F(tri[L_x=2, L_x=m, free], q)=(q-1)(q-2)^{2(m-1)+1}. \tag{15.1}$$

For the cases $L_y\geqslant 3$, we give the result in term of generating functions as shown from Eq. (9.3) to Eq. (9.6). For $F(tri[L_y=3, L_x=m, free], q)$ we calculate

$$\mathscr{N}(tri[L_y=3, free], q, z)$$
$$=(q-1)(q-2)^2\, z[(q-2)+(q-1)^3\, z-(q-1)^2\, z^2] \tag{15.2}$$

$$\mathscr{D}(tri[L_y=3, free], q, z)$$
$$=1-(q^4-7q^3+21q^2-33q+23)\, z+2(q-2)^2\, z^2. \tag{15.3}$$

There are thus $N_{F,\,tri,\,L_y=3,\,free,\,\lambda}=2$ terms,

$$\lambda_{tri,\,L_y=3,\,free,\,j}=\tfrac{1}{2}\,[q^4-7q^3+21q^2-33q+23\pm\sqrt{R_{t3f}}\,], \qquad j=1,2 \tag{15.4}$$

where

$$R_{t3f} = q^8 - 14q^7 + 91q^6 - 360q^5 + 949q^4 - 1708q^3 + 2047q^2 - 1486q + 497. \tag{15.5}$$

The locus $\mathscr{B}$ for the $L_x \to \infty$ limit of this strip is shown in Fig. 14. This locus is compact, consisting of two pairs of complex-conjugate arcs that do not separate the q plane into different regions and do not cross the real axis, so that no q_{cf} is defined. The eight endpoints of the arcs occur at the eight zeros of R_{t3f}, where the square root in (15.4) have branch point singularities. Thus, we find that the loci $\mathscr{B}$ for all of the lattice strips that we have considered with free longitudinal boundary conditions (which include the strips with free and cylindrical boundary conditions) are compact. These thus contrast with the behavior of $\mathscr{B}$ for many of the strips with periodic or twisted periodic longitudinal boundary conditions, for which $\mathscr{B}$ is noncompact.

We have also calculated the generating functions for $F(tri[L_y, L_x = m, free], q)$ with $L_y = 4, 5$ and find $N_{F,\, tri,\, L_y = 4,\, free,\, \lambda} = 6$ and $N_{F,\, tri,\, 5,\, free} = 13$. The results are too lengthy to include here. For the reader's convenience, we give the $L_y = 4$ results in the copy of this paper on the math-ph archive, and the $L_y = 5$ results are available from the authors. The respective loci $\mathscr{B}$

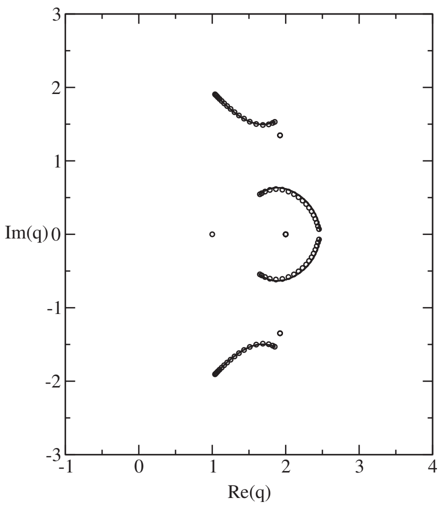

Fig. 14. Singular locus $\mathscr{B}$ in the q plane for $fl(tri, 3 \times \infty, free, q)$ for the $3 \times \infty$ strip of the triangular lattice with free boundary conditions. For comparison, zeros of the flow polynomial $F(tri, 3 \times L_x, free, q)$ for $L_x = 21$ are also shown.

for the $L_x \to \infty$ limits of these strips of the triangular lattice with $L_y = 4, 5$ and free boundary conditions again consist of arcs.

16. STRIPS OF THE TRIANGULAR LATTICE WITH CYLINDRICAL BOUNDARY CONDITIONS

We give the result in term of generating functions as shown from Eq. (9.3) to Eq. (9.6). For $F(tri[L_y = 2, L_x = m, cyl.], q)$ we have

$$\begin{aligned}&\mathcal{N}(tri[L_y = 2, cyl.], q, z)\\&\quad = (q-1)[1-(q-1)(q-2)(q-3)\, z]\end{aligned} \tag{16.1}$$

$$\begin{aligned}&\mathcal{D}(tri[L_y = 2, cyl.], q, z)\\&\quad = 1-(q^4-6q^3+15q^2-19q+11)\, z-(q-1)^3\,(q-2)\, z^2.\end{aligned} \tag{16.2}$$

The coefficients of the generating function for $F(tri, L_y = 3, L_x = m, cyl., q)$ are

$$A_{tri,\,3,\,cyl,\,0} = q-1 \tag{16.3}$$

$$A_{tri,\,3,\,cyl,\,1} = -(q-1)^2\,(2q^4-17q^3+58q^2-97q+68) \tag{16.4}$$

$$A_{tri,\,3,\,cyl,\,2} = -(q-1)^4\,(q-2)(q^3-3q^2-4q+13) \tag{16.5}$$

$$b_{tri,\,3,\,cyl,\,1} = -q^6+9q^5-36q^4+84q^3-127q^2+125q-65 \tag{16.6}$$

$$b_{tri,\,3,\,cyl,\,2} = (q-1)(q^7-11q^6+49q^5-110q^4+119q^3-24q^2-69q+52) \tag{16.7}$$

$$b_{tri,\,3,\,cyl,\,3} = -(q-1)^5\,(q-2)(q^2-2q-1). \tag{16.8}$$

We have also calculated the generating function for $F(tri[L_y = 4, L_x = m, cyl.], q)$. The results are too lengthy to present here, but we mention that

$$\deg \mathcal{D}(tri, L_y = 4, cyl.) = N_{F,\,tri,\,4,\,cyl} = 6. \tag{16.9}$$

17. $\mathcal{B}$ FOR 2D LATTICES

We can also obtain some results on $\mathcal{B}_{fl}$, in particular, q_{cf} values, for (infinite) 2D lattices. These follow directly from Eqs. (3.25). Using the relation (3.26) together with the result that $q_c(sq) = 3$,[86] we have

$$q_{cf}(sq) = 3. \tag{17.1}$$

Using (3.27) together with the result that $q_c(tri)=4,$[34] we have

$$q_{cf}(hc)=4. \tag{17.2}$$

For the honeycomb lattice, formally, $q_c=(1/2)(3+\sqrt{5}),$[36, 87] so that

$$q_{cf}(tri)=\frac{3+\sqrt{5}}{2}=2.6180\dots. \tag{17.3}$$

We note that two of these values of q_{cf} for 2D lattices can also be seen as complex-temperature roots of the equations for the phase transition points of the Potts model on these respective lattices. For example, for the square lattice, arguments have been given that the paramagnetic-antiferromagnetic (PM-AFM) transition occurs at a (physical) root of the equation $\zeta^2+4\zeta/\sqrt{q}+1=0$, where $\zeta=v/\sqrt{q}$.[88, 89] Evaluating this for the special case $\zeta=-\sqrt{q}$, i.e., $v=-q$, corresponding to the flow polynomial, yields $q=3$, the value of $q_{cf}(sq)$ in (17.1). For the triangular lattice, the paramagnetic-ferromagnetic (PM-FM) phase transition point is given by a (physical) root of the equation $\zeta^{-3}-3\zeta^{-1}-\sqrt{q}=0$;[90] evaluating this at $\zeta=-\sqrt{q}$ yields $q^2-3q+1=0$, the larger solution of which is $q_{cf}(tri)$ in (17.3). Since the honeycomb lattice is dual to the triangular lattice, the corresponding equation for the PM-FM phase transition point is obtained from that for the triangular lattice by the replacement $\zeta\to\zeta^{-1}$, i.e., $\zeta^3-3\zeta-\sqrt{q}=0$; again substituting $\zeta=-\sqrt{q}$ yields $(q-2)\sqrt{q}=0$. Although the root at $q=2$ of this equation is not the value of $q_{cf}(hc)$ as given in (17.2), it does correspond to a crossing of $\mathscr{B}$ as shown in Fig. 7. Note that this root of the $v=-q$ special case of the critical equation for the honeycomb lattice is equivalent, under duality, to a root at $q=2$ of the critical equation for the triangular lattice evaluated at $v=-1$, corresponding to the relation (3.27). Parenthetically, we recall that we observed that $\mathscr{B}_W$ passes through $q=2$ on infinite-length strips of the triangular lattice with periodic longitudinal boundary conditions.[47, 48, 59] Similar correspondences between complex-temperatures roots of the equations for Potts model phase transition points on 2D lattices and properties of $\mathscr{B}$ have been discussed, e.g., in refs. 75, 91–97. Since $q_c(kag)=3$, we finally have

$$q_{cf}(diced)=3. \tag{17.4}$$

As was discussed in our previous works on chromatic polynomials, from a study of the loci $\mathscr{B}_W$ for infinite-length strips of regular lattices of increasing widths (and with a variety of boundary conditions), one can formulate plausible inferences about the the boundary $\mathscr{B}_W$ for the limit of infinite width, i.e., the full 2D lattice. Given the duality relation (3.25), one can use either $\mathscr{B}_W$ or $\mathscr{B}_{fl}$, or both, for this purpose. This program is useful since one does not know $\mathscr{B}_W$ for any 2D lattice except for results on the triangular lattice (ref. 34, see, however, ref. 68). It was found in ref. 37 and subsequent papers that if one uses periodic longitudinal boundary conditions, then $\mathscr{B}_W$ exhibits a number of features that one would infer to hold for the 2D lattice, such as the property of separating the q plane into various regions, and crossing the real q axis at $q=0$ and a maximal value, q_c. The compactness of the loci $\mathscr{B}_W$ found for these strips is also the same as the compactness property of $\mathscr{B}_W$ for infinite regular lattices. This latter property is established by taking the $|V| \to \infty$ limit of a section of regular lattice Λ and applying the bound (3.29). This compactness is in accord with the fact, as discussed in refs. 42 and 43, that if $\mathscr{B}_W$ did extend infinitely far away from the origin, passing through $1/q=0$, then this point would be a point of nonanalyticity of the function $W(\Lambda, q)/q$, thereby precluding a Taylor-series expansion of this function around this point. However, well-known procedures exist for calculating Taylor-series expansions of $W(\Lambda, q)/q$ around $1/q=0$ for regular lattices.[25, 26]

In contrast, as discussed above, we find that the loci $\mathscr{B}_{fl}$ for the infinite-length limits of strips with periodic longitudinal boundary conditions are usually noncompact (an exception being the case of self-dual strips of the square lattice) and do not, in general, pass through $q=0$, although they do separate the q plane into various regions. From the duality relation (3.25) in conjunction with the bound (3.29), we infer that $\mathscr{B}_{fl}$ is compact for any infinite regular planar lattice Λ that has a dual planar lattice Λ^* (given the fact that a regular lattice has a fixed, finite degree for all of its vertices). In view of this result, there are thus two possibilities: (i) either $\mathscr{B}_{fl}$ will become compact for sufficiently great finite width for each type of infinite-length lattice strip, or (ii) we encounter another kind of noncommutativity in addition to (3.18), namely that $\lim_{L_y \to \infty} \mathscr{B}_{fl}(G_s, L_y \times \infty)$ is different from the accumulation set of the zeros of the flow polynomial of an $L_y \times L_x$ section of a regular lattice obtained by letting L_x and L_y both approach infinity with L_y/L_x a finite nonzero constant. Further study is needed to decide which of these two types of behavior occurs. We have already encountered the type of noncommutativity (ii) in our previous studies of $\mathscr{B}$ for the Potts model free energy on infinite-length, finite-width strips with periodic longitudinal boundary conditions; for these strips, $\mathscr{B}$ is noncompact in the v plane, reflecting the fact that the Potts model has a ferromagnetic critical point only at $T=0$ (i.e., $K=\infty$, hence $v=\infty$) for any width L_y, no matter how great, whereas for the 2D lattice defined in the usual thermodynamic limit $L_x \to \infty$, $L_y \to \infty$ with L_y/L_x equal to a finite nonzero constant, it has a ferromagnetic critical point at a finite temperature, so $\mathscr{B}$ is compact in the v plane.

Our present results show that the use of $\mathscr{B}_W$ and $\mathscr{B}_{fl}$ calculated on infinite-length finite-width strips are somewhat complementary. For example, the study of the loci $\mathscr{B}_W$ for finite-width, infinite-length strips of a lattice Λ with periodic longitudinal boundary conditions has the advantage that these loci are compact, as is true of $\mathscr{B}_W(\Lambda) = \mathscr{B}_{fl}(\Lambda^*)$ for the infinite dual pair of lattices Λ and Λ^*. On the other hand, the loci $\mathscr{B}_{fl}$ for all of the cyclic/Möbius strips of the square lattice for which we have calculated them, do exhibit the interesting feature of having q_{cf} equal to the value 3 for the infinite square lattice, whereas the q_c values for $\mathscr{B}_W$ on these strips only approach the square-lattice value asymptotically as L_y increases. Of course, for the self-dual families of planar strip graphs studied in refs. 63 and 75, $\mathscr{B}_{fl} = \mathscr{B}_W$ for each value of L_y separately, and in these cases, these loci share the common property of being compact and having $q_c = q_{cf} = 3$, the asymptotic value.

18. SUMMARY

Flow polynomials continue to be of considerable interest not just in statistical mechanics but also in mathematical graph theory and applied areas such as engineering. For a planar graphs G they are equivalent to chromatic polynomials on the respective dual graphs, via the relation (3.2). In this paper we have given exact calculations of flow polynomials $F(G, q)$ for lattice strips of various fixed widths and arbitrarily great lengths, with several different boundary conditions. We have determined the resultant functions fl giving the q-flows per face in the limit of infinite-length strips. We have also studied the zeros of $F(G, q)$ in the complex q plane and determined exactly the asymptotic accumulation set of these zeros $\mathscr{B}_{fl}$, across which fl is nonanalytic in the infinite-length limit. We found that these loci were noncompact for many strip graphs with periodic (or twisted periodic) longitudinal boundary conditions, in contrast to the usual behavior for the analogous loci $\mathscr{B}_W$ for the W function obtained from chromatic polynomials for these strips. We also found the interesting feature that, aside from the trivial case $L_y = 1$, the maximal point, q_{cf}, where $\mathscr{B}$ crosses the real axis, is universal on cyclic and Möbius strips of the square lattice for all widths for which we have calculated it and is equal to the asymptotic value $q_{cf} = 3$ for the infinite square lattice. Duality relations were used to derive a number of connections between fl and W, and $\mathscr{B}_{fl}$ and $\mathscr{B}_W$, for planar families of graphs. Since the flow polynomial $F(G, q)$ is, up to a factor, the special case of the Potts model partition function $Z(G, q, v)$ for $v = -q$, as given in (2.16), the study of the locus $\mathscr{B}_{fl}$ in the limit of infinitely many vertices gives insight into the singular locus for the free energy of the Potts model in the (q, v) plane.

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REFERENCES

1. W. T. Tutte, On the embedding of linear graphs in surfaces, *Proc. London Math. Soc. (2)* **51**:474–483 (1950). Sadly, we must record here that W. T. Tutte died on May 2, 2002, during the final days of the completion of this manuscript.
2. W. T. Tutte, On dichromatic polynomials, *J. Combin. Theory* **2**:301–320 (1967).
3. W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math.* **6**:80–91 (1954).
4. W. T. Tutte, On the algebraic theory of graph colourings, *J. Combin. Theory* **1**:15–50 (1966).
5. W. T. Tutte, *Graph Theory*, Encyclopedia of Mathematics and Its Applications, Vol. 21, G. C. Rota, ed. (Addison–Wesley, New York, 1984).
6. L. Ford and D. Fulkerson, *Flows in Networks* (Princeton University Press, Princeton, 1962).
7. F. Jaeger, Flows and generalized coloring theorems in graphs, *J. Combin. Theory Ser. B* **26**:205–216 (1979).
8. P. Seymour, Nowhere-zero 6-flows, *J. Combin. Theory Ser. B* **30**:130–135 (1981).
9. D. Younger, Integer flows, *J. Graph Theory* **7**:349–357 (1983).
10. F. Y. Wu, The Potts Model, *Rev. Modern Phys.* **54**:235–268 (1982).
11. J. W. Essam and C. Tsallis, The Potts model and flows: I. The pair correlation function, *J. Phys. A* **19**:409–422 (1986).
12. A. de Magalhaes and J. W. Essam, The Potts model and flows: II. Many-spin correlation functions, *J. Phys. A* **19**:1655–1679 (1986).
13. F. Y. Wu, Potts model and graph theory, *J. Stat. Phys.* **52**:99–112 (1988).
14. F. Jaeger, Nowhere-zero flow problems, in *Selected Topics in Graph Theory*, Vol. 3, L. W. Beineke and R. J. Wilson, eds. (Academic, New York, 1988), pp. 71–92.
15. A. de Magalhaes, J. W. Essam, and F. Y. Wu, Duality relation for Potts multispin correlation functions, *J. Phys. A* **23**:2651–2663 (1990).
16. G. Fan, Integer flows and cycle covers, *J. Combin. Theory Ser. B* **54**:113–122 (1992).
17. T. Brylawski and J. Oxley, The Tutte polynomial and its applications, in *Matroid Applications, Encyclopedia of Mathematics and Its Applications*, Chap. 6, Vol. 40, N. White, ed. (Cambridge University Press, Cambridge, 1992), pp. 123–225.
18. N. L. Biggs, *Algebraic Graph Theory*, 2nd ed. (Cambridge University Press, Cambridge, 1993).
19. D. J. A. Welsh, *Complexity: Knots, Colourings, and Counting*, London Math. Soc. Lecture Note Ser. 186 (Cambridge University Press, Cambridge, 1993).
20. T. R. Jensen and B. Toft, *Graph Coloring Problems* (Wiley, New York, 1995).

21. C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs* (Dekker, New York, 1997).
22. B. Bollobás, *Modern Graph Theory* (Springer, New York, 1998).
23. R. C. Read and E. G. Whitehead, Chromatic polynomials of homeomorphism classes of graphs, *Discrete Math.* **204**:337–356 (1999).
24. R. C. Read and E. G. Whitehead, The Tutte polynomial for homeomorphism classes of graphs, *Discrete Math.* **243**:267–272 (2002).
25. J. F. Nagle, A New subgraph expansion for obtaining coloring polynomials for graphs, *J. Combin. Theory Ser. B* **10**:42–59 (1971).
26. G. A. Baker, Linked-cluster expansion for the graph-vertex coloration problem, *J. Combin. Theory* **10**:217 (1971).
27. N. L. Biggs, R. M. Damerell, and D. A. Sands, Recursive families of graphs, *J. Combin. Theory Ser. B* **12**:123–131 (1972).
28. S. Beraha and J. Kahane, Is the four-color conjecture almost false, *J. Combin. Theory Ser. B* **27**:1–12 (1979).
29. S. Beraha, J. Kahane, and N. Weiss, Limits of chromatic zeros of some families of maps, *J. Combin. Theory Ser. B* **28**:52–65 (1980).
30. S. Beraha, J. Kahane, and N. Weiss, Limits of zeros of recursively defined families of polynomials, *Proc. Nat. Acad. Sci. U.S.A.* **72**:4209 (1975).
31. S. Beraha, J. Kahane, and N. Weiss, Limits of zeros of recursively defined families of polynomials. Studies in Foundations and Combinatorics, *Adv. Math., Supplementary Studies* **1**:213–232 (1978).
32. A. Sokal, Chromatic zeros are dense in the whole complex plane, cond-mat/0012369.
33. R. C. Read, A large family of chromatic polynomials, in *Proc. 3rd Caribbean Conference on Combinatorics and Computing*, C. C. Cadogan, ed. (University of the West Indies Press, Barbados, 1981), pp. 23–41 (1981).
34. R. J. Baxter, Chromatic polynomials of large triangular lattices, *J. Phys. A* **20**:5241–5261 (1987).
35. R. C. Read and G. F. Royle, in *Graph Theory, Combinatorics, and Applications*, Vol. 2, Y. Alavi *et al.*, eds. (Wiley, New York, 1991), pp. 1009–1029.
36. J. Salas and A. Sokal, Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem, *J. Stat. Phys.* **86**:551–579 (1997).
37. R. Shrock and S.-H. Tsai, Asymptotic limits and zeros of chromatic polynomials and ground state entropy of Potts antiferromagnets, *Phys. Rev. E* **55**:5165–5179 (1997).
38. R. Shrock and S.-H. Tsai, Families of graphs with chromatic zeros lying on circles, *Phys. Rev. E* **56**:1342–1345 (1997).
39. M. Roček, R. Shrock, and S.-H. Tsai, Chromatic polynomials for families of strip graphs and their asymptotic limits, *Phys. A* **252**:505–546 (1998).
40. R. Shrock and S.-H. Tsai, Ground state entropy of Potts antiferromagnets on homeomorphic families of strip graphs, *Phys. A* **259**:315–348 (1998).
41. M. Roček, R. Shrock, and S.-H. Tsai, Chromatic polynomials on $J(\prod H)\,I$ strip graphs and their asymptotic limits, *Phys. A* **259**:367–387 (1998).
42. R. Shrock and S.-H. Tsai, Families of graphs with $W_r(\{G\}, q)$ functions that are nonanalytic at $1/q = 0$, *Phys. Rev. E* **56**:3935–3943 (1997).
43. R. Shrock and S.-H. Tsai, Ground state degeneracy of Potts antiferromagnets: Homeomorphic classes with noncompact W boundaries, *Phys. A* **265**:186–223 (1999).
44. R. Shrock and S.-H. Tsai, Ground state degeneracy of Potts antiferromagnets: Cases with noncompact W boundaries having multiple points at $1/q = 0$, *J. Phys. A* **31**:9641–9665 (1998).
45. R. Shrock and S.-H. Tsai, Ground state entropy of the Potts antiferromagnet on cyclic strip graphs, *J. Phys. A (Letts.)* **32**:L195–L200 (1999).

46. R. Shrock and S.-H. Tsai, Ground state entropy of Potts antiferromagnets on cyclic polygon chain graphs, *J. Phys. A* **32**:5053–5070 (1999).
47. R. Shrock and S.-H. Tsai, Ground state degeneracy of Potts antiferromagnets on 2D lattices: Approach using infinite cyclic strip graphs, *Phys. Rev. E* **60**:3512–3515 (1999).
48. R. Shrock and S.-H. Tsai, Exact partition functions for Potts antiferromagnets on cyclic lattice strips, *Phys. A* **275**:429–449 (2000).
49. N. L. Biggs, Matrix method for chromatic polynomials, *J. Combin. Theory Ser. B* **82**: 19–29 (2001).
50. R. Shrock, $T=0$ partition functions for Potts antiferromagnets on Möbius strips and effects of graph topology, *Phys. Lett. A* **261**:57–62 (1999).
51. N. L. Biggs and R. Shrock, $T=0$ partition functions for Potts antiferromagnets on square lattice strips with (twisted) periodic boundary conditions, *J. Phys. A (Letts.)* **32**: L489–L493 (1999).
52. R. Shrock, Chromatic polynomials and their zeros and asymptotic limits for families of graphs, in Proceedings of the 1999 British Combinatorial Conference, *Discrete Math.* **231**:421–446 (2001).
53. A. Sokal, Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions, *Combin. Probab. Comput.* **10**:41–77 (2001).
54. S.-C. Chang and R. Shrock, Ground state entropy of the Potts antiferromagnet with next-nearest-neighbor spin–spin couplings on strips of the square lattice, *Phys. Rev. E* **62**: 4650–4664 (2000).
55. S.-C. Chang and R. Shrock, Ground state entropy of the Potts antiferromagnet on strips of the square lattice, *Phys. A* **290**:402–430 (2001).
56. J. Salas and A. Sokal, Transfer matrices and partition-function zeros for antiferromagnetic Potts models I. General theory and square-lattice chromatic polynomial, *J. Stat. Phys.* **104**:609–699 (2001).
57. S.-C. Chang and R. Shrock, Structural properties of Potts model partition functions and chromatic polynomials for lattice strips, *Phys. A* **296**:131–182 (2001).
58. S.-C. Chang and R. Shrock, $T=0$ partition functions for Potts antiferromagnets on lattice strips with fully periodic boundary conditions, *Phys. A* **292**:307–345 (2001).
59. S.-C. Chang and R. Shrock, Ground state entropy of the Potts antiferromagnet on triangular lattice strips, *Ann. Phys.* **290**:124–155 (2001).
60. J. Jacobsen and J. Salas, Transfer matrices and partition-function zeros for antiferromagnetic Potts models II. Extended results for square-lattice chromatic polynomial, *J. Stat. Phys.* **104**:701–723 (2001).
61. J. Salas and R. Shrock, Exact $T=0$ partition functions for Potts antiferromagnets on sections of the simple cubic lattice, *Phys. Rev. E* **64**:011111 (2001).
62. S.-C. Chang, Chromatic polynomials for lattice strips with cyclic boundary conditions, *Phys. A* **296**:495–522 (2001).
63. S.-C. Chang and R. Shrock, Potts model partition functions for self-dual families of graphs, *Phys. A* **301**:301–329 (2001).
64. N. L. Biggs, Chromatic polynomials and representations of the symmetric group, Algebraic Graph Theory Workshop, Edinburgh (July 2001), LSE report LSE-CDAM-01-02 (2001).
65. N. L. Biggs, M. Klin, and P. Reinfeld, Algebraic methods for chromatic polynomials, Algebraic Graph Theory Workshop, Edinburgh (July, 2001), LSE report LSE-CDAM-01-06 (2001).
66. S.-C. Chang, Exact chromatic polynomials for toroidal chains of complete graphs, *Phys. A* **313**:397–426 (2002).
67. S.-C. Chang and R. Shrock, General structural results for Potts model partition functions on lattice strips, *Phys. A* **316**:335–379 (2002).

68. J. Jacobsen, J. Salas, and A. Sokal, Transfer matrices and partition-function zeros for antiferromagnetic Potts models III. Triangular-lattice chromatic polynomial, cond-mat/0204587.
69. S.-C. Chang and R. Shrock, Tutte polynomials and related asymptotic limiting functions for recursive families of graphs, *Adv. in Appl. Math.*, in press (math-ph/0112061).
70. R. Shrock, Exact Potts model partition functions on ladder graphs, *Phys. A* **283**:388–446 (2000).
71. S.-C. Chang and R. Shrock, Exact Potts model partition functions on wider arbitrary-length strips of the square lattice, *Phys. A* **296**:234–288 (2001).
72. S.-C. Chang and R. Shrock, Exact Potts model partition functions on strips of the triangular lattice, *Phys. A* **286**:189–238 (2000).
73. S.-C. Chang and R. Shrock, Exact Potts model partition functions on strips of the honeycomb lattice, *Phys. A* **296**:183–233 (2001).
74. S.-C. Chang and R. Shrock, Exact partition function for the Potts model with next-nearest neighbor couplings on strips of the square lattice, *Internat. J. Modern Phys. B* **15**:443–478 (2001).
75. S.-C. Chang and R. Shrock, Complex-temperature phase diagrams for the q-state Potts model on self-dual families of graphs and the nature of the $q \to \infty$ limit, *Phys. Rev. E* **64**:066116 (2001).
76. S.-C. Chang, J. Salas, and R. Shrock, Exact Potts model partition function for strips of the square lattice, *J. Stat. Phys.* **107**:1207–1253 (2002).
77. C. Fortuin and P. Kasteleyn, On the random cluster model. I. Introduction and relation to other models, *Physica* **57**:536–564 (1972).
78. K. Appel and W. Haken, Every map is four colorable, part I: Discharging, *Illinois J. Math.* **21**:429–490 (1977).
79. K. Appel, W. Haken, and J. Koch, Every map is four colorable, part II: Reducibility, *Illinois J. Math.* **21**:491–567 (1977).
80. B. Grünbaum and G. Shephard, *Tilings and Patterns* (Freeman, New York, 1987).
81. R. Shrock and S.-H. Tsai, Lower bounds and series for the ground state entropy of the Potts antiferromagnet on Archimedean lattices and their duals, *Phys. Rev. E* **56**:4111–4124 (1997).
82. N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/.
83. F. Bernhart, Catalan, Motzkin, and Riordan numbers, *Discrete Math.* **204**:73–112 (1999).
84. R. Shrock and S.-H. Tsai, Upper and lower bounds for ground state entropy of antiferromagnetic Potts models, *Phys. Rev. E* **56**:2733–2737 (1997).
85. R. Shrock and S.-H. Tsai, Ground state entropy of Potts antiferromagnets and the approach to the 2d thermodynamic limit, *Phys. Rev. E* **58**:4332–4339 (1998), cond-mat/9808057.
86. A. Lenard, unpublished, as credited in E. H. Lieb, Residual entropy of square ice, *Phys. Rev.* **162**:162–171 (1967).
87. R. Shrock and S.-H. Tsai, Ground state entropy and the $q = 3$ Potts antiferromagnet on the honeycomb lattice, *J. Phys. A* **30**:495–500 (1997).
88. R. J. Baxter, Critical antiferromagnetic square-lattice Potts model, *Proc. Roy. Soc. London Ser. A* **383**:43–54 (1982).
89. The results obtained in ref. 88 involved a mapping from the Potts model to a six-vertex model. A complication pertaining to a mismatch between the boundary conditions for the Potts and vertex model was noted there. See also H. Saleur, Zeros of chromatic polynomials: A new approach to Beraha conjecture using quantum groups, *Commun. Math. Phys.* **132**:657–679 (1990); and H. Saleur, The antiferromagnetic Potts model in two

dimensions: Berker–Kadanoff phase, antiferromagnetic transition, and the role of Beraha numbers, *Nuclear Phys. B* **360**:219–263 (1991).

90. D. Kim and R. Joseph, Exact transition temperature of the Potts model with q states per site for the triangular and honeycomb lattices, *J. Phys. C* **7**:L167–L169 (1974).
91. P. Martin and J. M. Maillard, Zeros of the partition function for the triangular lattice 3-state Potts model, *J. Phys. A* **19**:L547–L551 (1986).
92. C. N. Chen, C. K. Hu, and F. Y. Wu, Partition function zeros of the square lattice Potts model, *Phys. Rev. Lett.* **76**:169–172 (1996).
93. F. Y. Wu, G. Rollet, H. Y. Huang, J. M. Maillard, C. K. Hu, and C. N. Chen, Directed compact lattice animals, restricted partitions of an integer, and the infinite-state Potts model, *Phys. Rev. Lett.* **76**:173–176 (1996).
94. V. Matveev and R. Shrock, Complex-temperature singularities in Potts models on the square lattice, *Phys. Rev. E* **54**:6174–6185 (1996).
95. H. Feldmann, R. Shrock, and S.-H. Tsai, A mapping relating complex and physical temperatures in the 2D q-state Potts model and some applications, *J. Phys. A (Lett.)* **30**:L663–668 (1997).
96. H. Feldmann, R. Shrock, and S.-H. Tsai, Complex-temperature partition function zeros of the Potts model on the honeycomb and kagomé lattices, *Phys. Rev. E* **57**:1335–1346 (1998).
97. H. Feldmann, A. J. Guttmann, I. Jensen, R. Shrock, and S.-H. Tsai, Study of the Potts model on the honeycomb and triangular lattices: Low-temperature series and partition function zeros, *J. Phys. A* **31**:2287–2310 (1998).